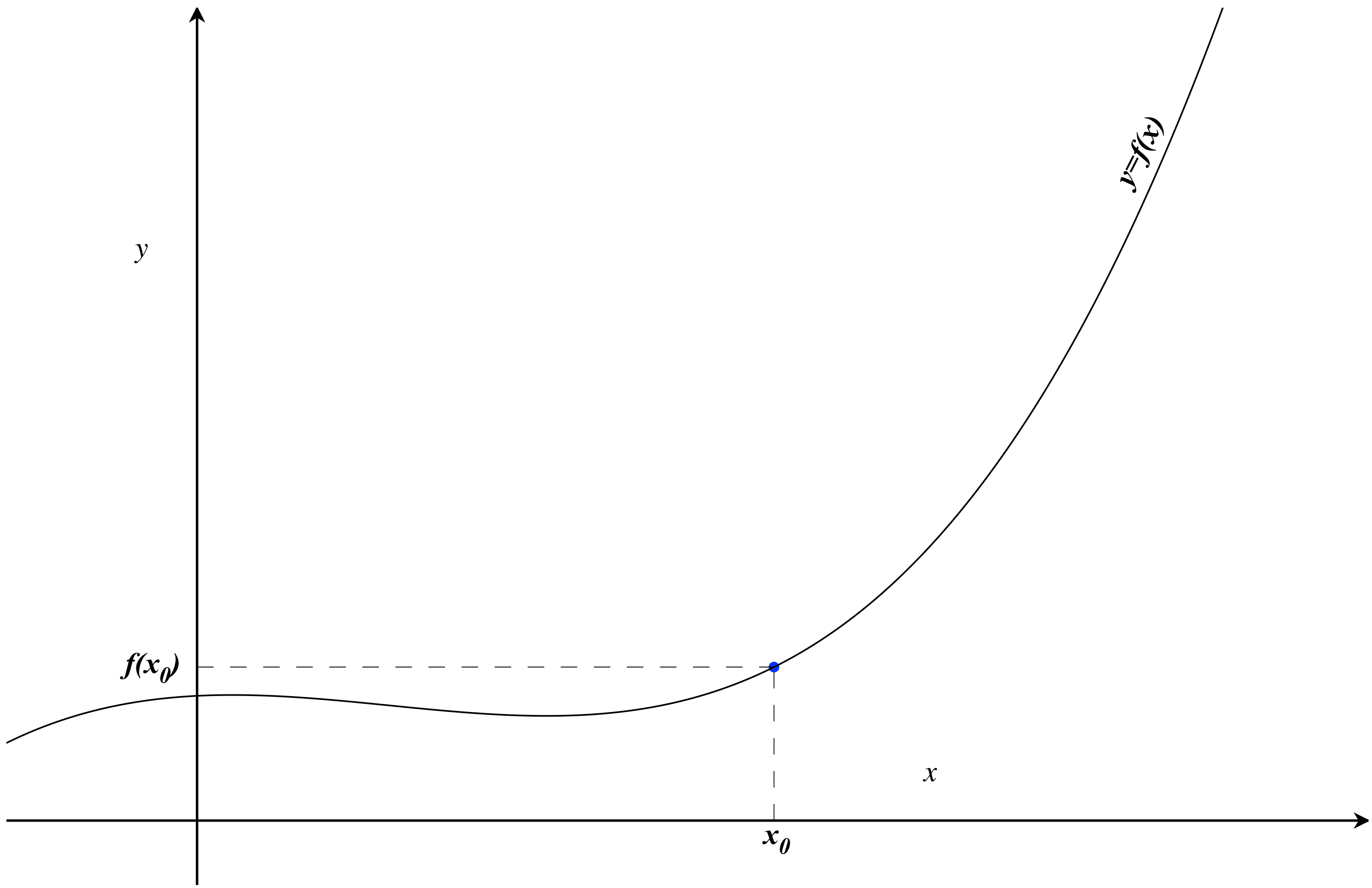
**Question:** How do we find the slope of the graph $y = f(x)$ at a point $(x_0, f(x_0))$ on the graph?



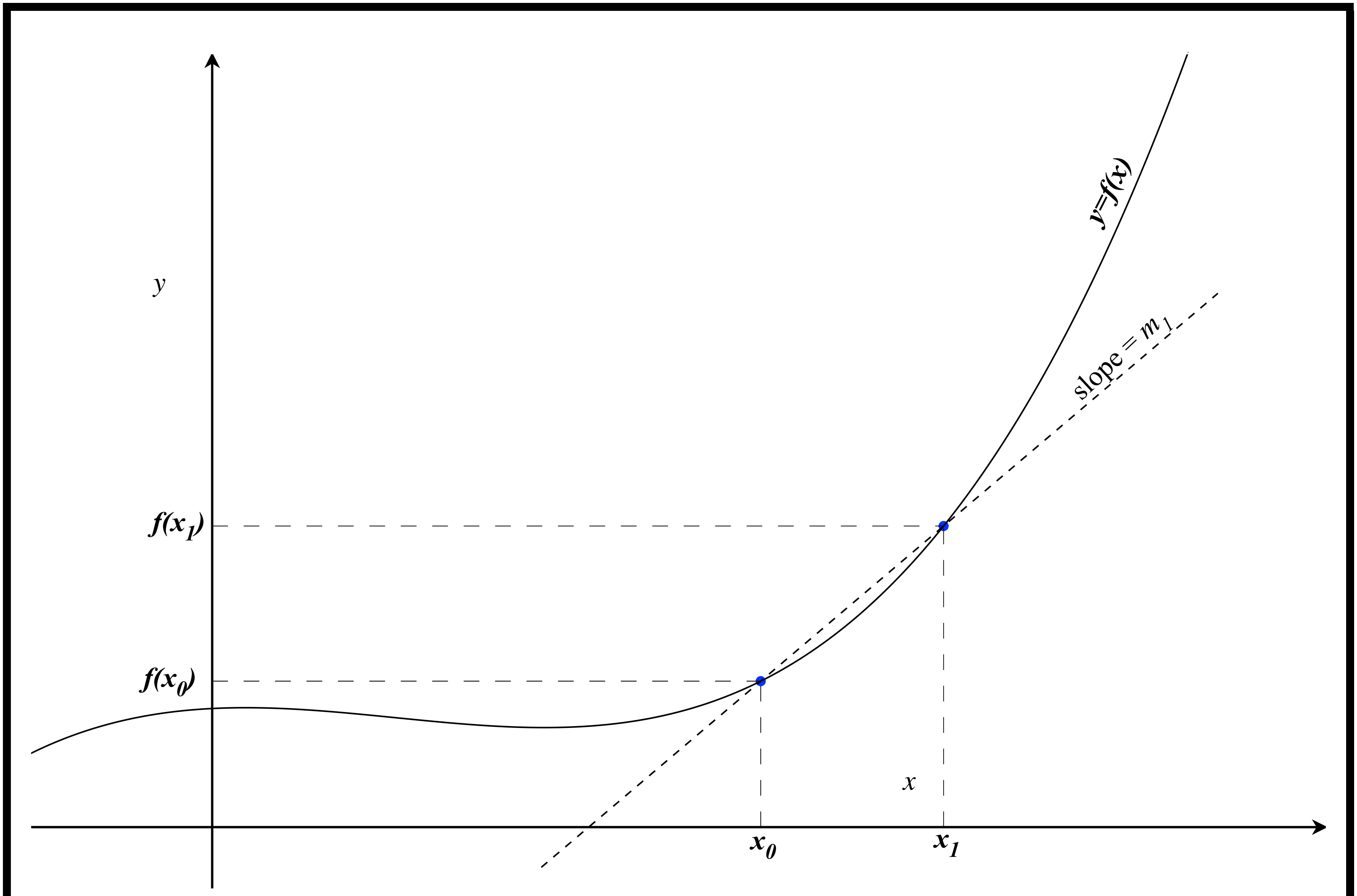
**Step 1.** *Find an approximation to the slope.*

**Problem:** we need two points to calculate a slope but we have only one point: $(x_0, f(x_0))$.

**Solution:** choose another point on the graph, say $(x_1, f(x_1))$, and calculate the slope of the (secant) line that connects these two points:

$$m_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The slope $m_1$ is an approximation of the (unknown) slope that we seek.

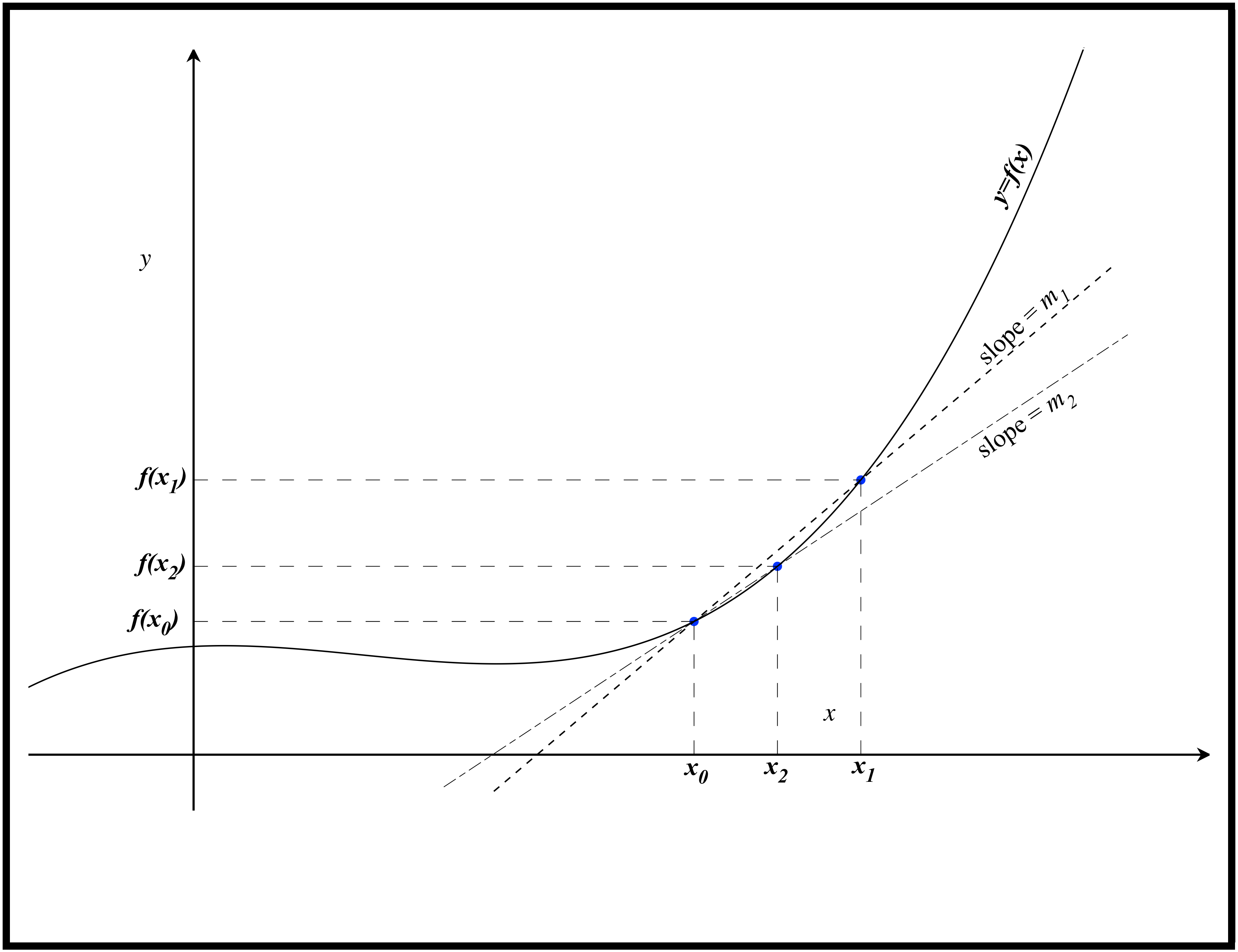

y=f(x)
slope = $m_1$
y
$f(x_1)$
$f(x_0)$
x
$x_0$
$x_1$

**Step 2.** *Find a (repeatable) way to improve the approximation.*

**Intuition:** If we choose a point $x_2$ that is closer to $x_0$ than $x_1$, and find the slope

$$m_2 = \frac{f(x_2) - f(x_0)}{x_2 - x_0}$$

of the line connecting $(x_0, f(x_0))$ to $(x_2, f(x_2))$, then $m_2$ will be a better approximation to the slope ***at*** $(x_0, f(x_0))$ than $m_1$.


y=f(x)
y
slope = m1
slope = m2
f(x1)
f(x2)
f(x0)
x
x0
x2
x1

**Step 3 and beyond:** *Repeat step 2 and take a limit.*

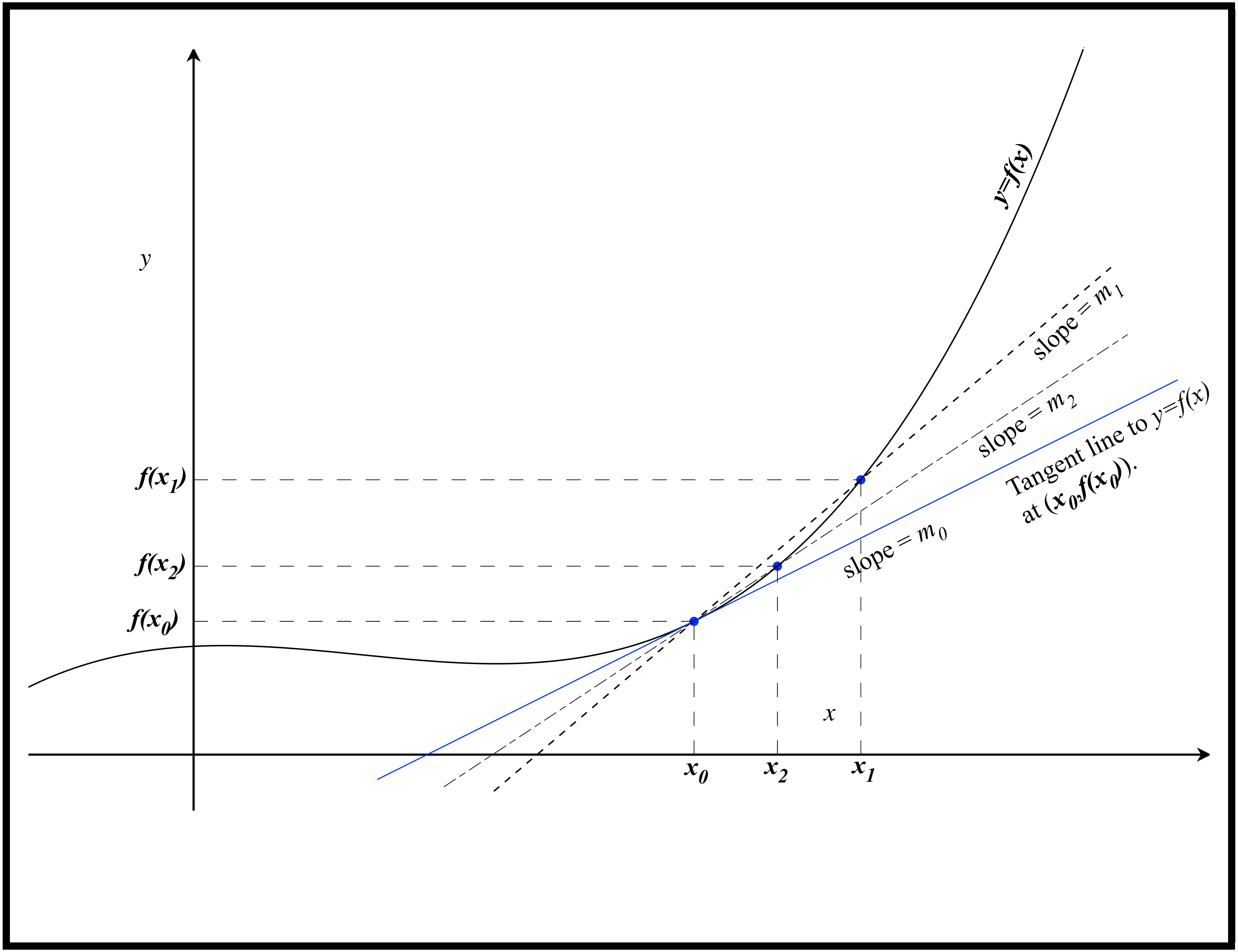
**Intuition:** If we continue to choose points on the graph that are closer and closer to $(x_0, f(x_0))$ and compute the slopes of the secant lines connecting these points to $(x_0, f(x_0))$, then these slopes should approach the slope ***at*** $(x_0, f(x_0))$, ***if it exists.***

**Definition:** *The slope of the graph $y = f(x_0)$ at the point $(x_0, f(x_0))$ is given by*

$$m_0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

***if the limit exists****. If the limit does not exist, then the graph does not have a slope at that point.*

**Definition:** *If the graph $y = f(x)$ has a slope $m_0$ at $(x_0, f(x_0))$, then the straight line that passes through $(x_0, f(x_0))$ with slope $m_0$ is called the* ***tangent line*** *to the graph at that point.*



y=f(x)
y
slope = $m_1$
slope = $m_2$
Tangent line to $y$=$f(x)$
at ($x_0$,$f(x_0)$).
slope = $m_0$
$f(x_1)$
$f(x_2)$
$f(x_0)$
x
$x_0$
$x_2$
$x_1$

**Example 1:** Find the slope of the graph $y = x^2$ at the point $(1, 1)$, and find the equation of the tangent line to $y = x^2$ at $(1, 1)$.

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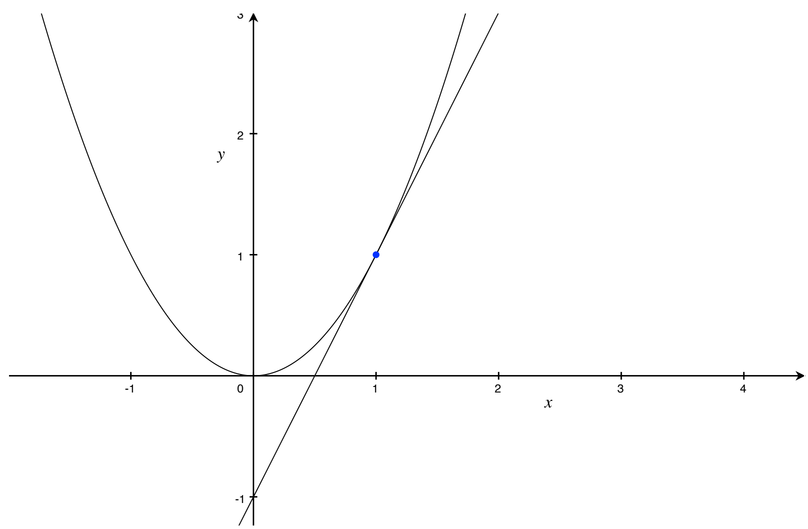
$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1} x + 1 = 2.$$

The tangent line we seek passes through the point $(1, 1)$ with slope $m = 2$... To find its equation, use the ***point-slope*** formula:

$$y - y_0 = m_0(x - x_0) \implies y - 1 = 2(x - 1) \implies y = 2x - 1.$$



**Terminology and Notation:**

The *slope* of $y = f(x)$ at $(x_0, f(x_0))$ is called the ***derivative*** of $y = f(x)$ ***at*** $x_0$, and denoted by $f'(x_0)$ or $y'(x_0)$. I.e.,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

if the limit exists.

(*) The equation of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ can be written

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{or} \quad y = f(x_0) + f'(x_0)(x - x_0).$$

(*) If we write $x = x_0 + h$, then $x - x_0 = h$.

(*) In this case, $x \to x_0$ is the same as $h \to 0$.

(*) We can (re)write the definition of the derivative as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

**Example 1, revisited:** The derivative of $y = x^2$ at $x = 1$ is

$$\begin{aligned} y'(1) &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{\not{1} + 2h + h^2 - \not{1}}{h} \\ &= \lim_{h \to 0} \frac{\not{h}(2+h)}{\not{h}} = \lim_{h \to 0} 2 + h = 2. \end{aligned}$$

**Example 2:** Find the derivative of $y = x^2$ at the points $x = 2$ and $x = -1$.

$$\begin{aligned} y'(2) &= \lim_{h\to 0} \frac{(2+h)^2 - 4}{h} \\ &= \lim_{h\to 0} \frac{\not{4} + 4h + h^2 - \not{4}}{h} \\ &= \lim_{h\to 0} \frac{\not{h}(4+h)}{\not{h}} = \lim_{h\to 0} 4 + h = 4. \end{aligned}$$

and

$$\begin{aligned} y'(-1) &= \lim_{h\to 0} \frac{(-1+h)^2 - 1}{h} \\ &= \lim_{h\to 0} \frac{\not{1} - 2h + h^2 - \not{1}}{h} \\ &= \lim_{h\to 0} \frac{\not{h}(-2+h)}{\not{h}} = \lim_{h\to 0} -2 + h = -2. \end{aligned}$$

**Definition:** The derivative of the function $y = f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

at every point $x$ where the limit exists.

**Terminology:** If $f'(x)$ is defined at a point $x_0$, then the function $f(x)$ is said to be *differentiable* at $x_0$. If $f(x)$ is differentiable at every point $x$ in some interval $I = (a, b)$, then $f(x)$ is differentiable in $I$.

**Example 3:** Find the derivative of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x. \end{aligned}$$

This limit exists for all $x$, so $f(x) = x^2$ is differentiable on the entire real line.

**Example 4.** Find the derivative of the function $l(x) = ax + b$.

$$\begin{aligned} l'(x) &= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} \\ &= \lim_{h \to 0} \frac{\cancel{ax} + ah + \cancel{b} - \cancel{ax} - \cancel{b})}{h} \\ &= \lim_{h \to 0} \frac{a\cancel{h}}{\cancel{h}} = \lim_{h \to 0} a = a \end{aligned}$$

**Observation:** The graph $y = l(x)$ is a straight line with slope $a$, so the fact that $l'(x) = a$ shows that the derivative, as defined, extends the concept of slope from straight lines to more general curves.

**Comment:** We don't have to use $h$ to denote the change in $x$. Another (perhaps better) symbol for the change in $x$ is $\Delta x$, pronounced *delta x*.

**Example 5:** Find the derivative of $g(x) = x^{1/2}$, and determine the interval(s) where the function is differentiable.

$$
\begin{aligned}
g'(x) &= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{1/2} - x^{1/2}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[ \left( \frac{(x+\Delta x)^{1/2} - x^{1/2}}{\Delta x} \right) \cdot \overbrace{\left( \frac{(x+\Delta x)^{1/2} + x^{1/2}}{(x+\Delta x)^{1/2} + x^{1/2}} \right)}^{=1} \right] \\
&= \lim_{\Delta x \to 0} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x \left( (x+\Delta x)^{1/2} + x^{1/2} \right)} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x} \left( (x+\Delta x)^{1/2} + x^{1/2} \right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{(x+\Delta x)^{1/2} + x^{1/2}} = \frac{1}{2x^{1/2}}
\end{aligned}
$$

$g'(x) = \dfrac{1}{2x^{1/2}}$ is defined for all $x > 0$, i.e., $g(x) = x^{1/2}$ is differentiable in the interval $(0, \infty)$.

*Let's collect the results from the examples thus far...*

- $f(x) = x^2 \implies f'(x) = 2x \implies \left(x^2\right)' = 2 \cdot x^1$
- $l(x) = x \implies l'(x) = 1 \implies \left(x^1\right)' = 1 \cdot x^0$
- $g(x) = x^{1/2} \implies g'(x) = \dfrac{1}{2x^{1/2}} \implies \left(x^{1/2}\right)' = \dfrac{1}{2}x^{-1/2}$
- $f(x) = \dfrac{1}{x^2} \implies f'(x) = -\dfrac{2}{x^3} \implies \left(x^{-2}\right)' = -2 \cdot x^{-3}$

**Observation:** The derivatives of these functions all satisfy the rule

$$\left(x^k\right)' = kx^{k-1}.$$

**Notation:** It is common to use the notation $\dfrac{d}{dx}$ for the operation of differentiation—finding a derivative. In other words, we write

$$\frac{d}{dx}f(x) = f'(x).$$

**Power rule:** If $k$ is any real number, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

**Comment:** This rule is relatively easy to justify if $k$ is a positive integer (as done in the book). It is not much harder to do if $k$ is a negative integer. When $k$ is a *rational* number (not an integer) the justification is more challenging, and the most challenging case is when $k$ is irrational.

**More basic rules...**

***Constant functions:***

$$\frac{d}{dx}C = 0$$

*Because*

$$\frac{d}{dx}C = \lim_{h\to 0}\frac{C-C}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

**(*)** This agrees with the fact that the graph of a constant function is a horizontal line, and the slope of a horizontal line is 0.

***Sums and differences:***

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x).$$

*Because* (for sums, differences are handled analogously)

$$\begin{aligned}
\frac{d}{dx}(f(x) + g(x)) &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\
&= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h} \\
&= \lim_{h \to 0} \left( \frac{f(x+h) - f(x)}{h} + \frac{g(x+h)) - g(x)}{h} \right) \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h)) - g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

**Example 7:**

$$\frac{d}{dx}\left(x^2 + \frac{1}{x}\right) = \frac{d}{dx}\left(x^2 + x^{-1}\right) = 2x + (-1) \cdot x^{-2} = 2x - x^{-2}.$$

***Constant multiples:***

$$\frac{d}{dx}(Cf(x)) = Cf'(x).$$

*Because*

$$\begin{aligned}\frac{d}{dx}(Cf(x)) &= \lim_{h\to 0} \frac{Cf(x+h) - Cf(x)}{h} \\ &= \lim_{h\to 0} C\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= C \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = Cf'(x)\end{aligned}$$

**Example 8:** Find the derivative of $y = 2x^3 - 3x^2 + 4x - 5$.

$$\begin{aligned}y' &= \frac{d}{dx}(2x^3) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(4x) - \frac{d}{dx}(5) \\ &= 2\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) - \frac{d}{dx}(5) \\ &= 2(3x^2) - 3(2x) + 4(1) - 0 = 6x^2 - 6x + 4\end{aligned}$$

**Example 9:** Find the derivative of $f(x) = \dfrac{3x^2 - 1}{5\sqrt{x}}$.

First, simplify the $f(x)$ and rewrite it as a difference of multiples of powers:

$$f(x) = \frac{3x^2 - 1}{5\sqrt{x}} = \frac{3x^2}{5x^{1/2}} - \frac{1}{5x^{1/2}} = \frac{3}{5}x^{3/2} - \frac{1}{5}x^{-1/2}$$

Then differentiate

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{3}{5}x^{3/2}\right) - \frac{d}{dx}\left(\frac{1}{5}x^{-1/2}\right) \\ &= \frac{3}{5} \cdot \frac{d}{dx}\left(x^{3/2}\right) - \frac{1}{5} \cdot \frac{d}{dx}\left(x^{-1/2}\right) \\ &= \frac{3}{5} \cdot \frac{3}{2}x^{1/2} - \frac{1}{5} \cdot \left(-\frac{1}{2}\right) x^{-3/2} \\ &= \frac{9}{10}x^{1/2} + \frac{1}{10}x^{-3/2} \end{aligned}$$