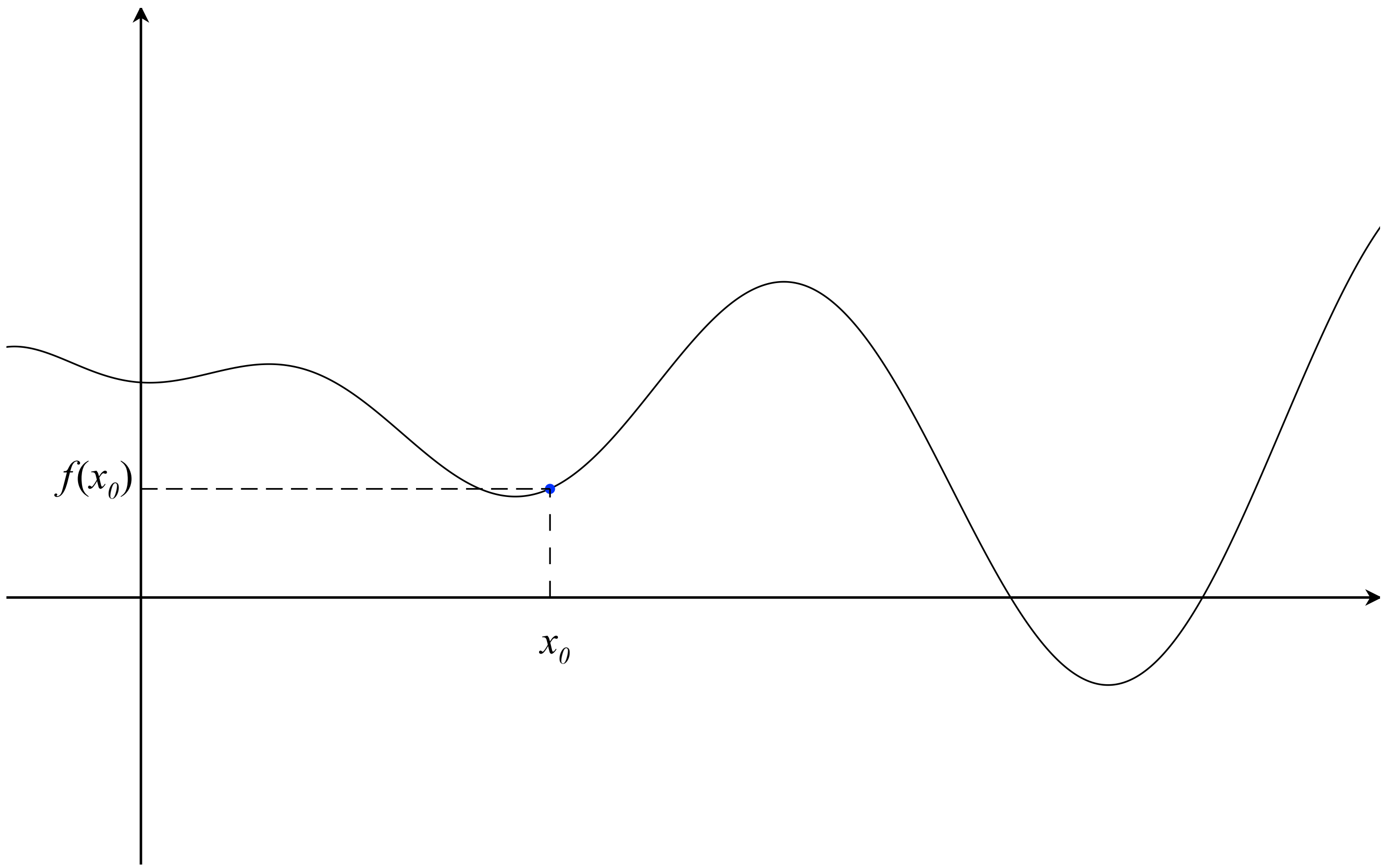
The graph of $y = f(x)$:



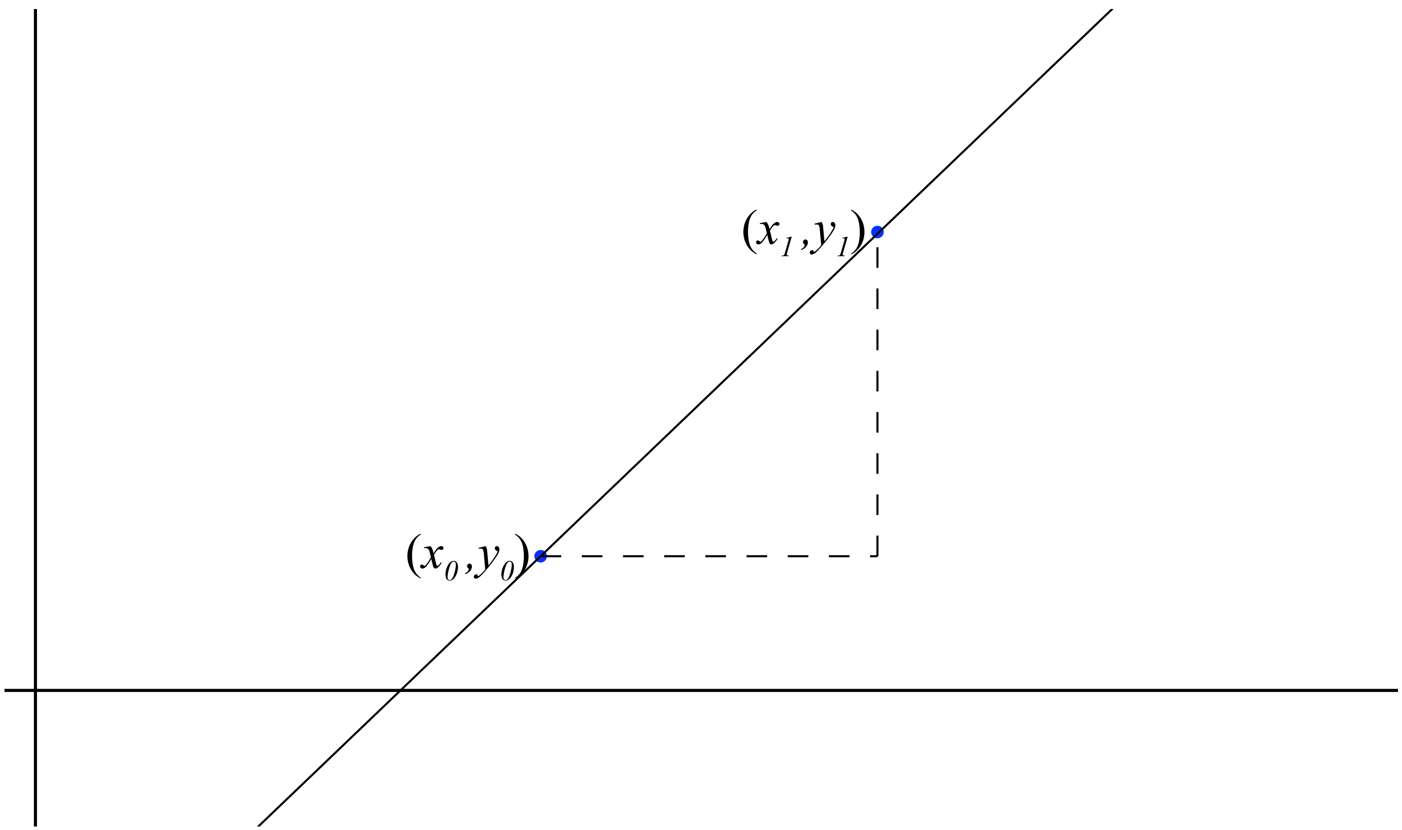
Question: What is the *direction* of the graph at the point $(x_0, f(x_0))$?

We describe the direction of a ***straight line*** by its slope:

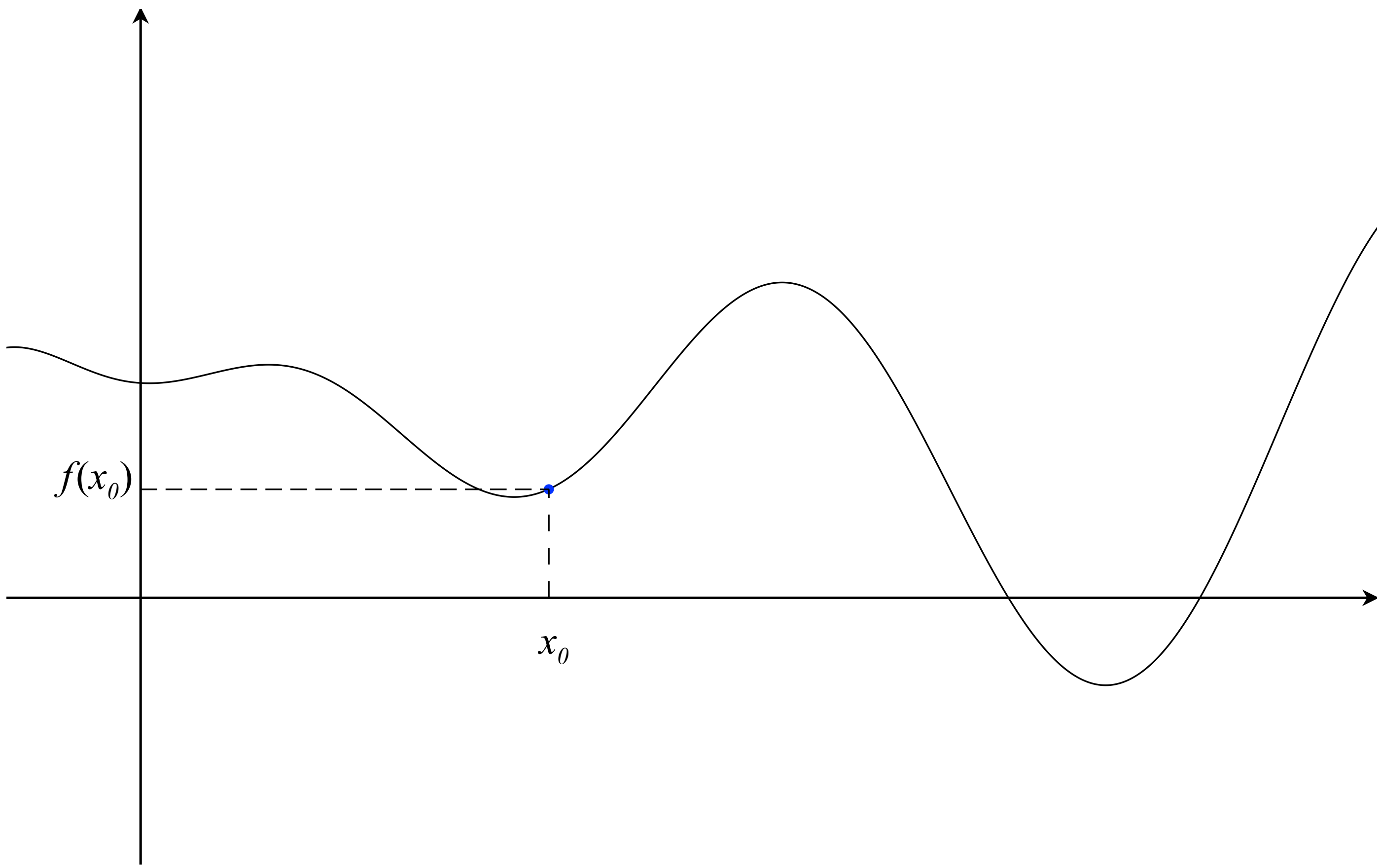
Definition: the slope of a *straight line* is defined by the *difference quotient*

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

where (x_0, y_0) and (x_1, y_1) are points on the line satisfying $x_1 \neq x_0$.

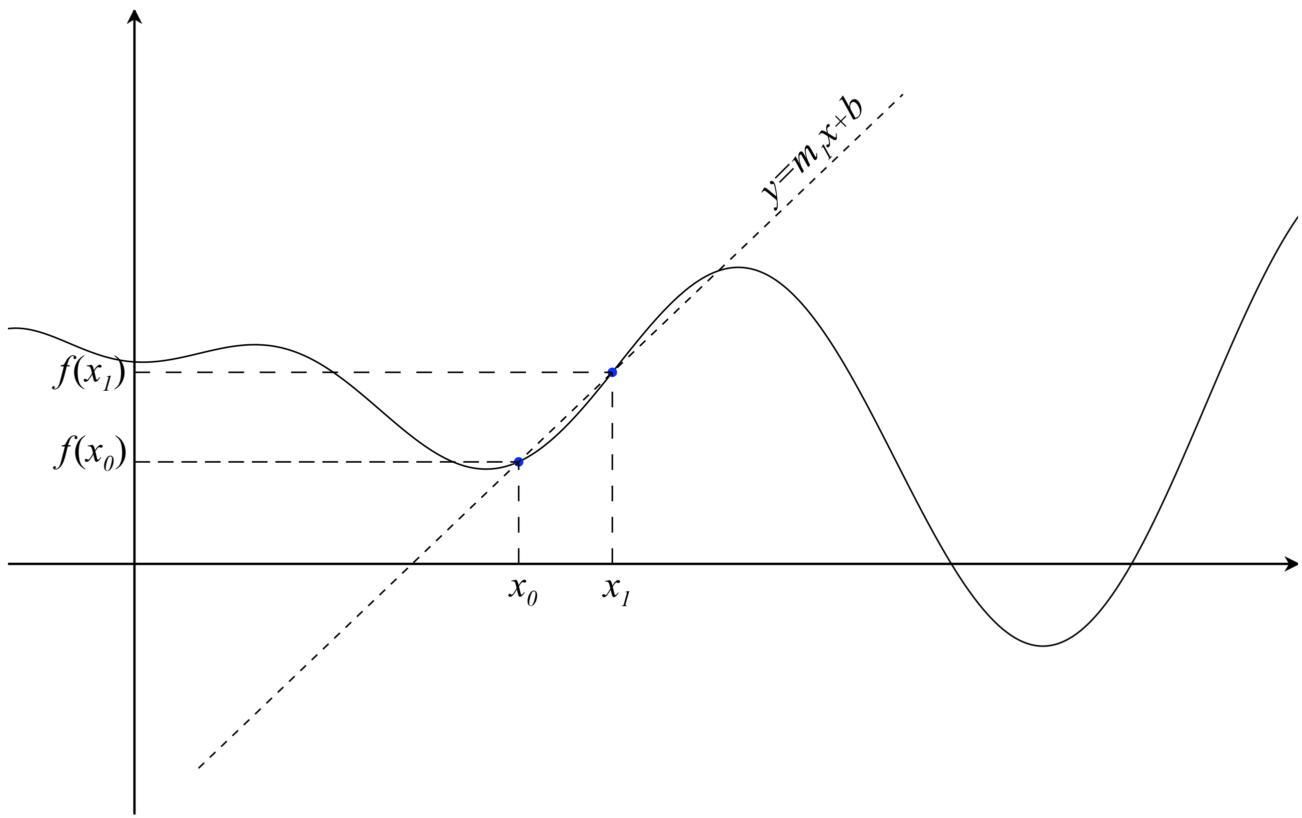


The problem we encounter when we try to define the slope of $y = f(x)$ at the point $(x_0, f(x_0))$ is that we only have one point to work with.



The solution to this problem is to use the following three steps...

(i) Find an approximate value for the slope at the given point: Find a second point on the graph $y = f(x)$, and use the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ to compute a slope:



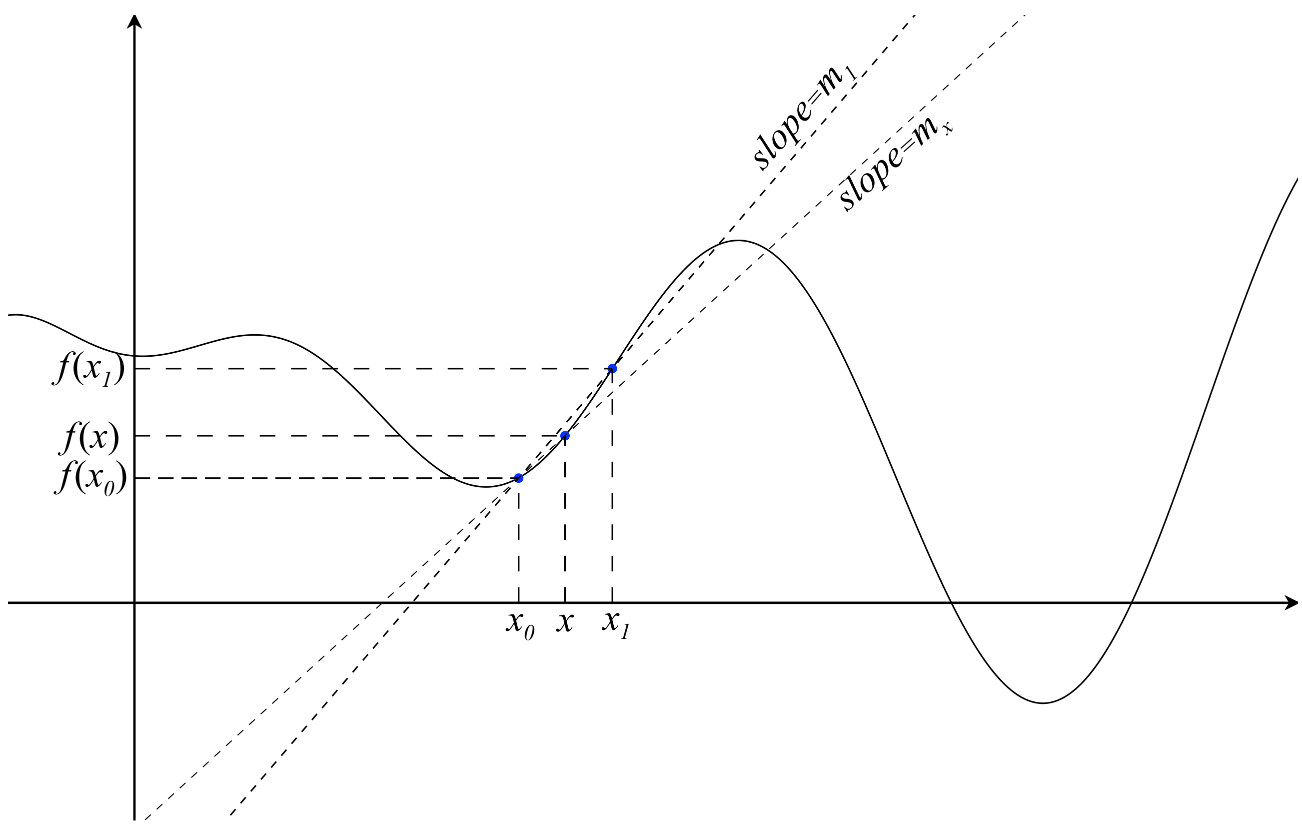
Approximate slope:

$$m_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

(ii) Improve the approximation, by repeating step (i) with points on the graph that are getting closer and closer to $(x_0, f(x_0))$. I.e., compute

$$m_x = \frac{f(x) - f(x_0)}{x - x_0}$$

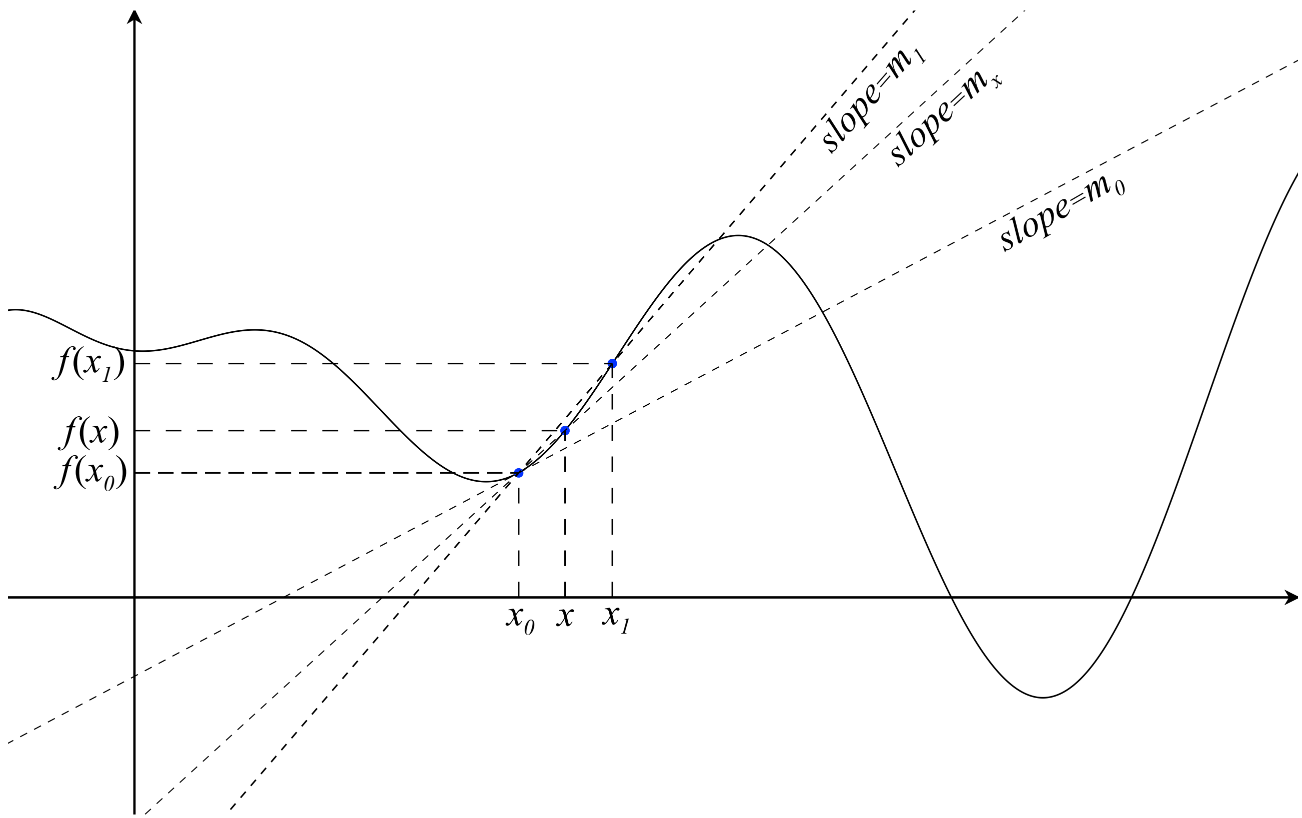
for values of x that are getting closer and closer to x_0.



(iii) Take a limit of these approximations, ***if it exists***. I.e., compute

$$m_0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If the limit exists, then we declare m_0 to be the slope of $y = f(x)$ at the point $(x_0, f(x_0))$.



Example: Suppose that $f(x) = x^3$ and $x_0 = 1$, then $f(x_0) = 1^3 = 1$. The limit that defines the slope of the graph $y = x^3$ at the point $(1, 1)$ is

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x-1}} = \lim_{x\to 1} x^2 + x + 1 = 3.$$

I.e., the slope of the graph $y = x^3$ at the point $(1, 1)$ is $m_0 = 3$.