Observation: The derivative $f'(x)$ gives the slope (direction) of the graph $y = f(x)$ at each point $(x, f(x))$ on the graph.

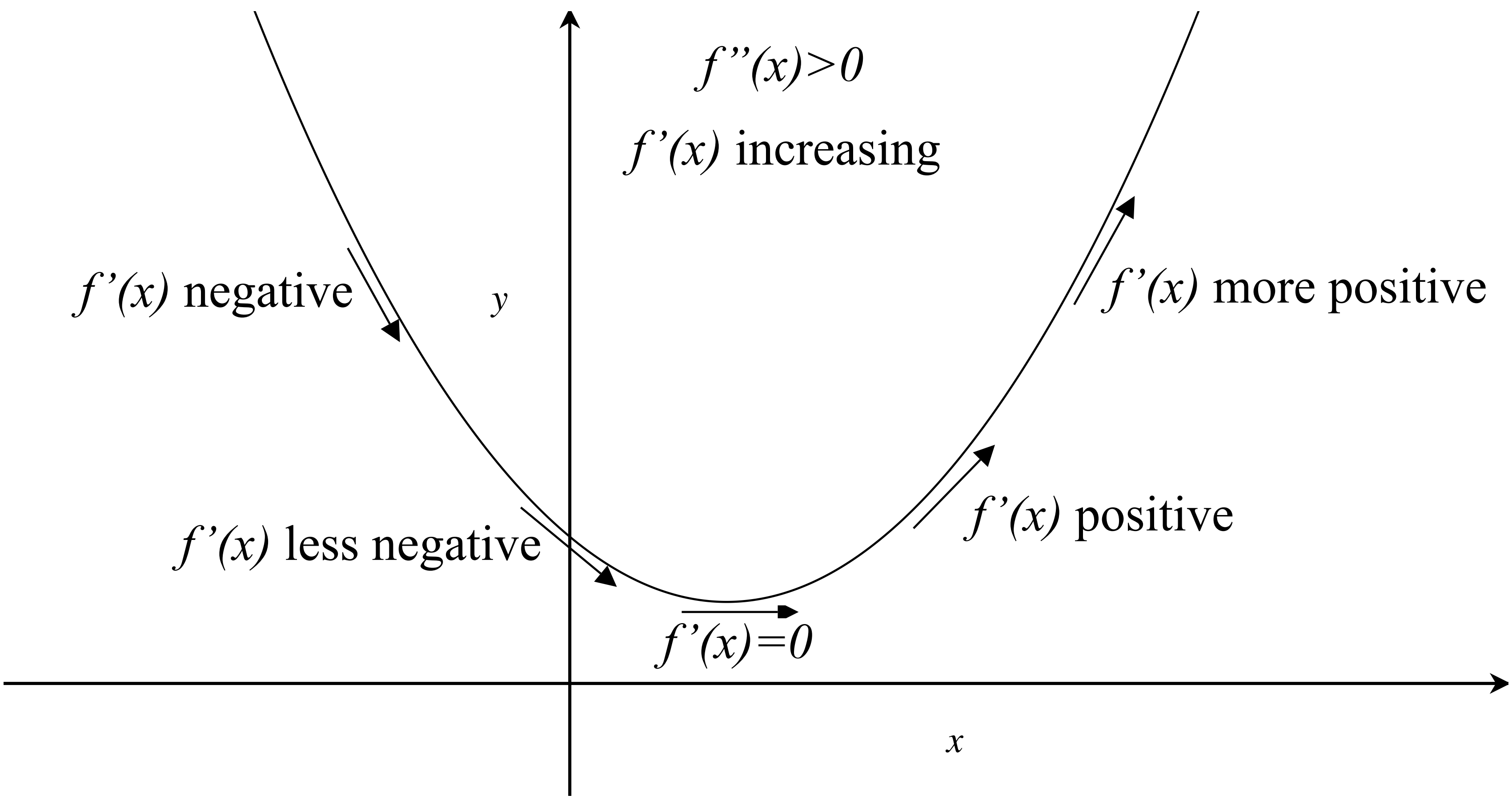
(*) The second derivative $f''(x)$ gives the *rate of change* of the first derivative $f'(x)$.

(*) This means that $f''(x)$ describes *how the slope of the graph is changing*, i.e., $f''(x)$ describes how the graph is *curving*.

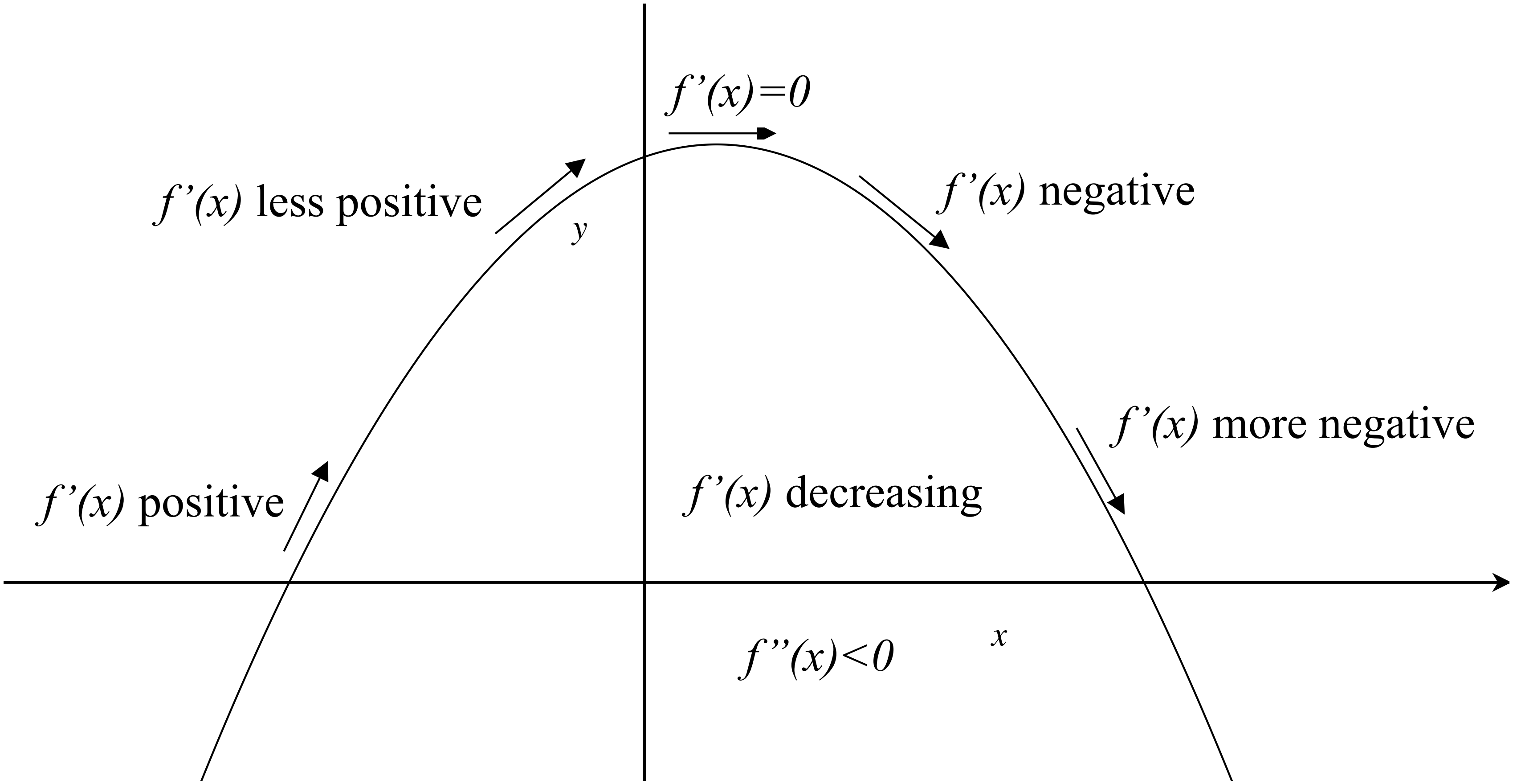
Specifically:

- If $f''(x) > 0$, then $f'(x)$ is increasing, so the slope of $y = f(x)$ is increasing and the graph is *curving up*.
- If $f''(x) < 0$, then $f'(x)$ is decreasing, so the slope of $y = f(x)$ is decreasing and the graph is *curving down*.

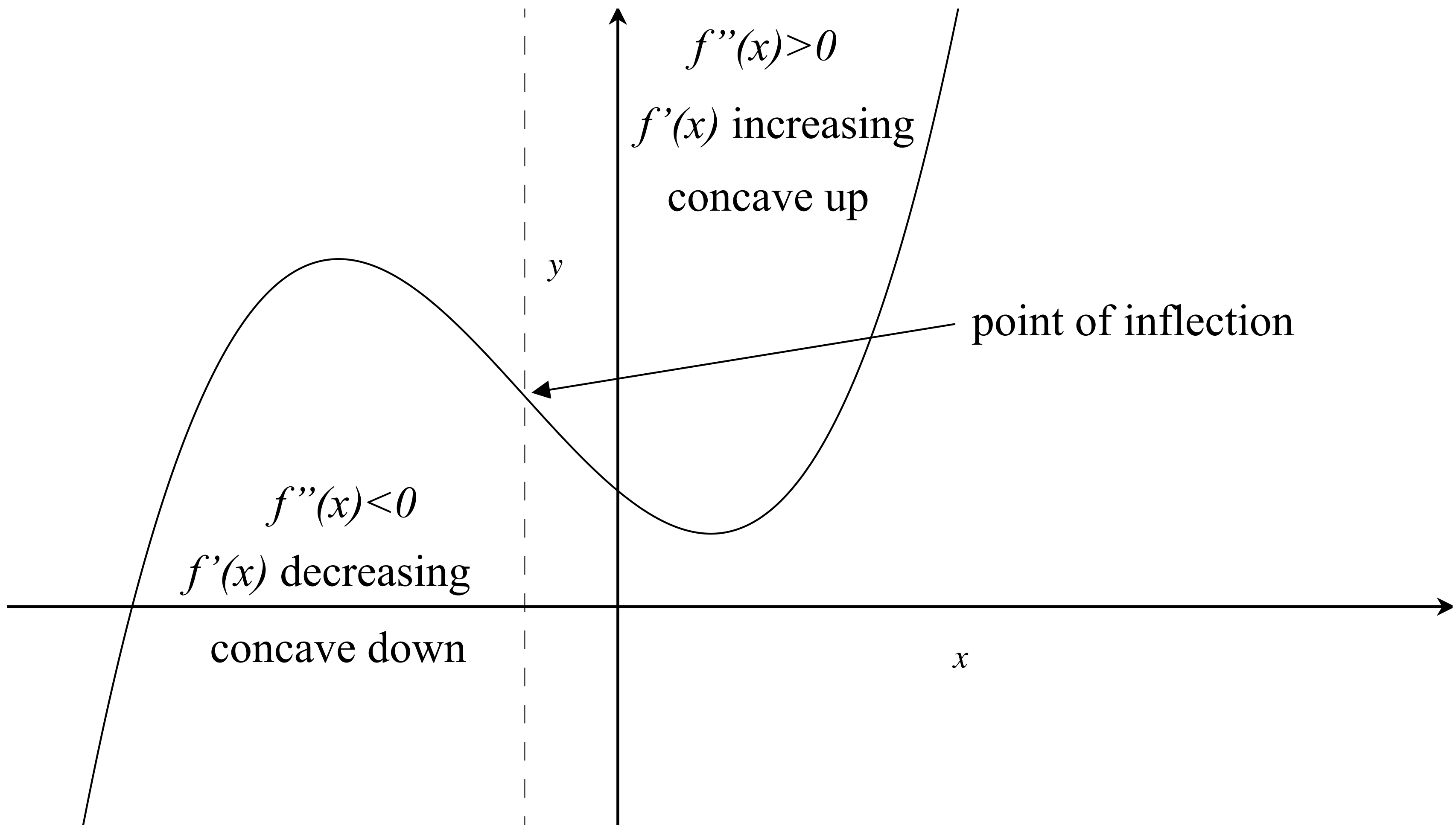
The proper term for 'curving up' is ***convex*** or ***concave up***.



The proper term for ‘curving down’ is ***concave*** or ***concave down***.



A point on the graph where the concavity changes is called a ***point of inflection***



Example: Find the points of inflection on the graph

$$y = \frac{1}{2}x^4 - 5x^3 + 12x^2 + 6x + 7.$$

Comment: The concavity changes when y'' changes ***sign***. This can only happen at points where $y'' = 0$ or at points where y'' is undefined.

Step 1. Find *possible* points of inflection...

$$y' = 2x^3 - 15x^2 + 24x + 6 \implies y'' = 6x^2 - 30x + 24$$

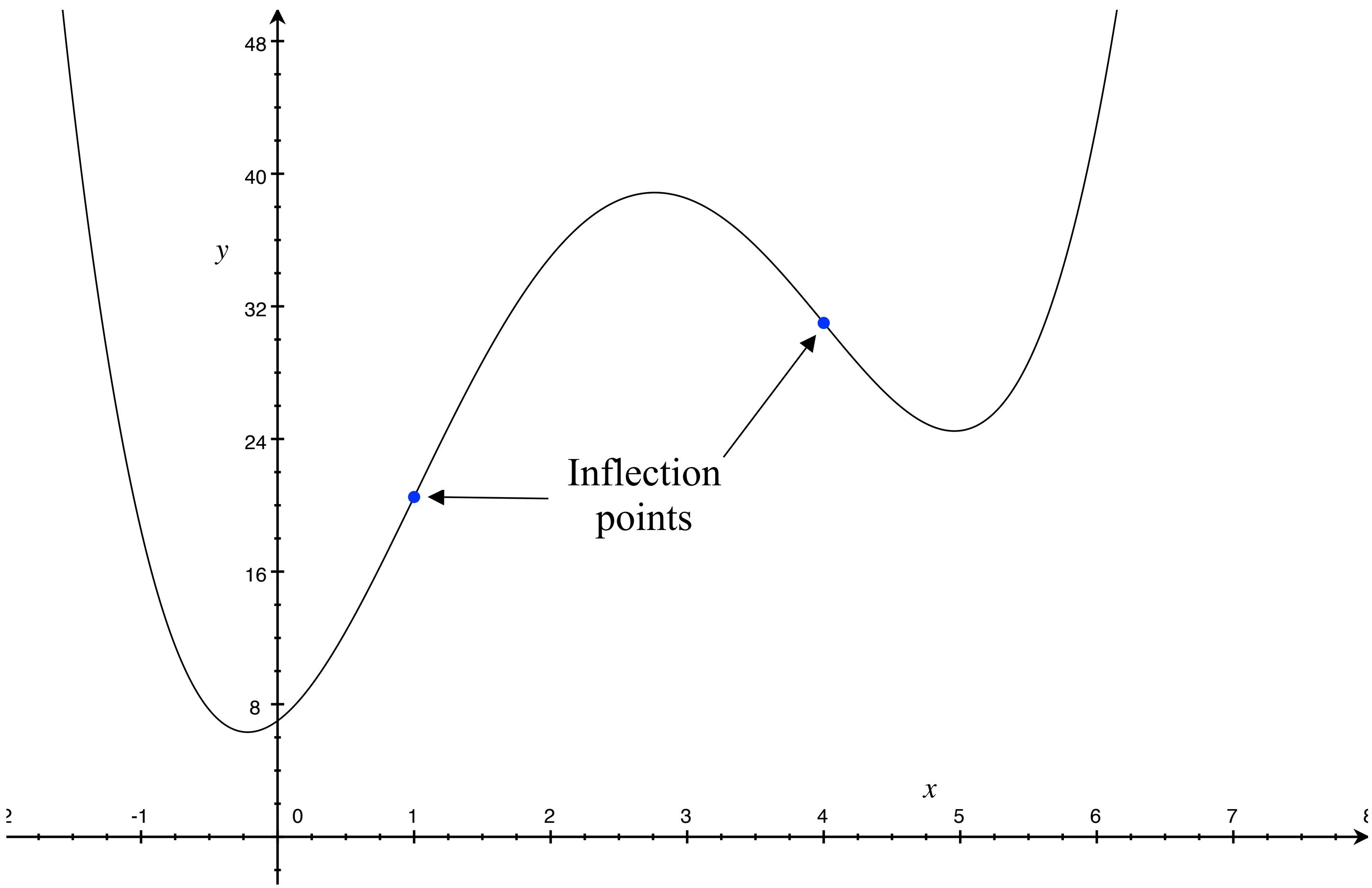
$$y'' = 0 \implies 6(x-1)(x-4) = 0 \implies x = 1 \text{ or } x = 4.$$

So there are *possible* points of inflection at $(1, y(1)) = (1, 20.5)$ and $(4, y(4)) = (4, 31)$.

Step 2. Analysis:

- If $x < 1$, then $y''(x) = 6(x-1)(x-4) = (+) \cdot (-) \cdot (-) = (+)$
- If $1 < x < 4$, then $y''(x) = 6(x-1)(x-4) = (+) \cdot (+) \cdot (-) = (-)$
- If $4 < x$, then $y''(x) = 6(x-1)(x-4) = (+) \cdot (+) \cdot (+) = (+)$

Conclusion: The graph is concave up for $x < 1$, concave down for $1 < x < 4$ and concave up for $4 < x$, and both points, $(1, 20.5)$ and $(4, 31)$, ***are*** inflection points.



Graph of $y = \frac{1}{2}x^4 - 5x^3 + 12x^2 + 6x + 7$

Example. Find the points of inflection on the graph $y = 4x^2e^{-0.5x}$.

(*) $y' = 8xe^{-0.5x} + 4x^2e^{-0.5x} \cdot (-0.5) = e^{-0.5x}(8x - 2x^2)$.

(*) $y'' = (-0.5) \cdot e^{-0.5x}(8x - 2x^2) + e^{-0.5x}(8 - 4x) = e^{-0.5x}(x^2 - 8x + 8)$

(*) Possible inflection points: $y'' = 0 \implies x^2 - 8x + 8 = 0$

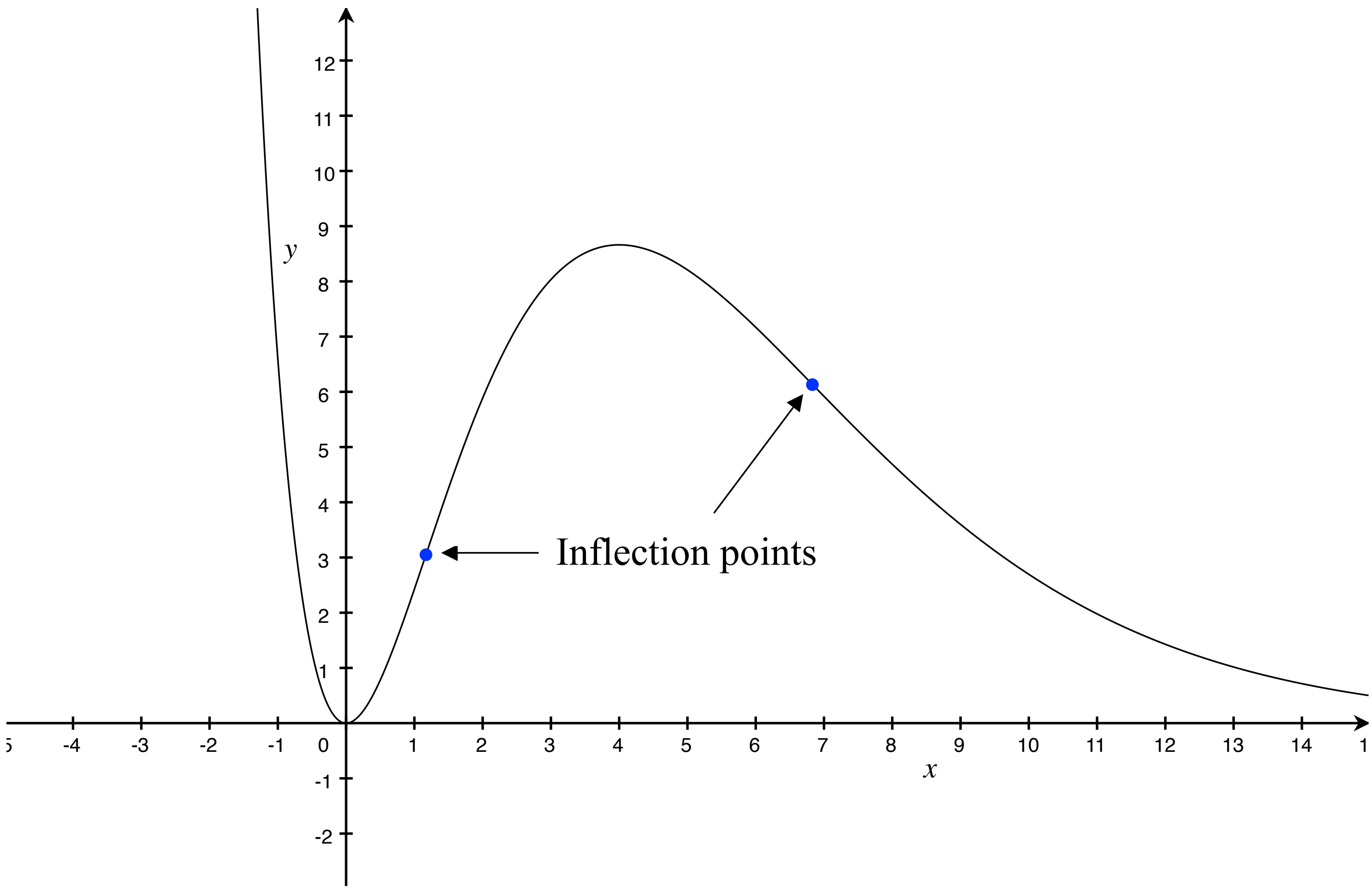
$$\implies x = \frac{8 \pm \sqrt{64 - 32}}{2}$$

$$\implies x_1 = 4 - \sqrt{8} \approx 1.17 \text{ and } x_2 = 4 + \sqrt{8} \approx 6.83.$$

(*) Analysis: y'' can only change at (possible) inflection points...

$$\left.\begin{array}{rcl} y''(1) & = & e^{-0.5} > 0 \\ y''(4) & = & -8e^{-2} < 0 \end{array}\right\} (x_1, y_1) \approx (1.17, 3.05) \text{ is an inflection point}$$

$$\left.\begin{array}{rcl} y''(4) & = & -8e^{-2} < 0 \\ y''(8) & = & 8e^{-4} > 0 \end{array}\right\} (x_2, y_2) \approx (6.83, 6.13) \text{ is an inflection point}$$



Graph of $y = 4x^2e^{-0.5x}$.

Second derivative test, a second explanation.

The second derivative test says that if $f'(x^*) = 0$ and...

(*) ... $f''(x^*) > 0$, then $f(x^*)$ is a local minimum value.

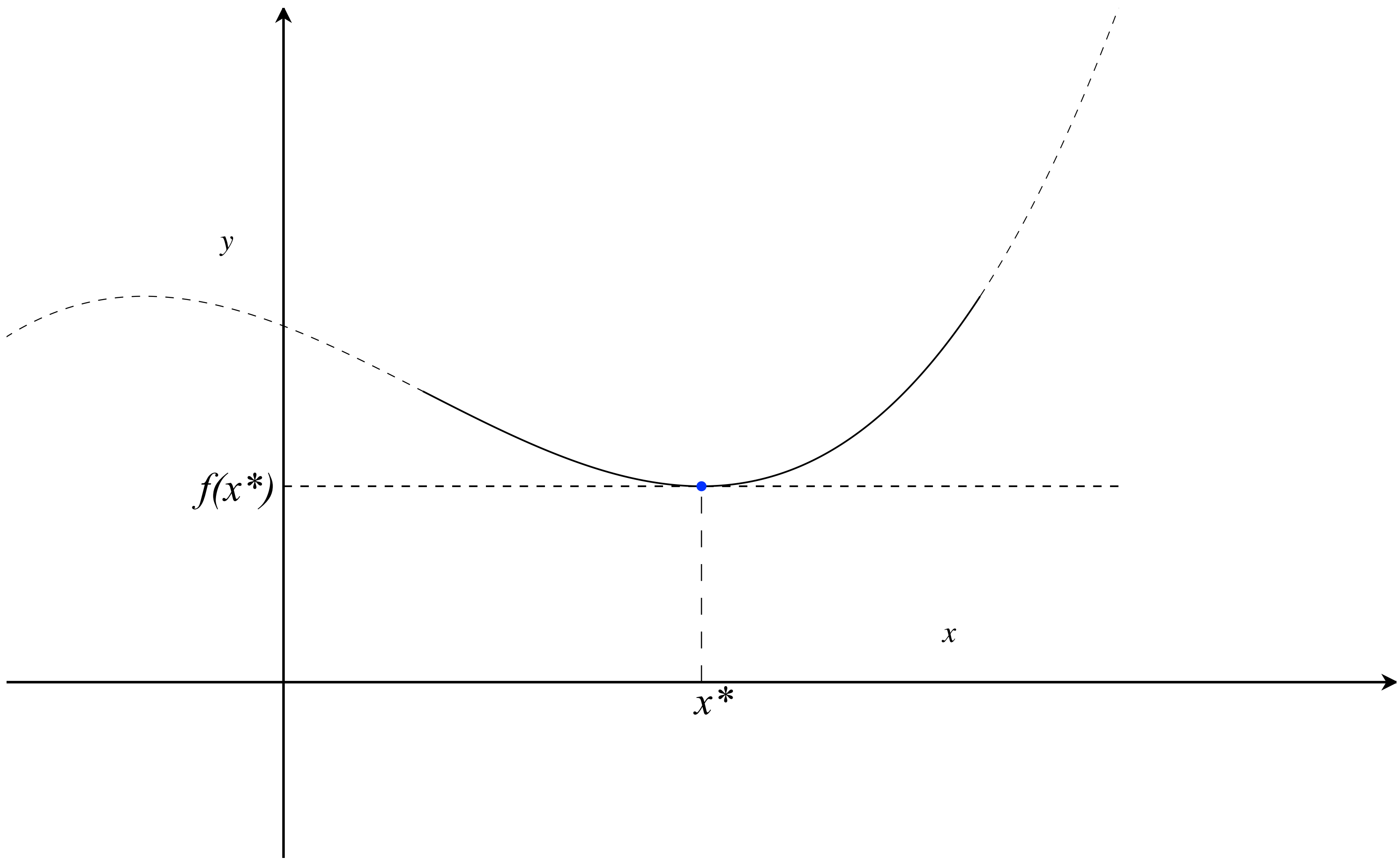
(*) ... $f''(x^*) < 0$, then $f(x^*)$ is a local maximum value.

We can (also) explain this test in terms of the concavity of the graph around the critical point x^*.

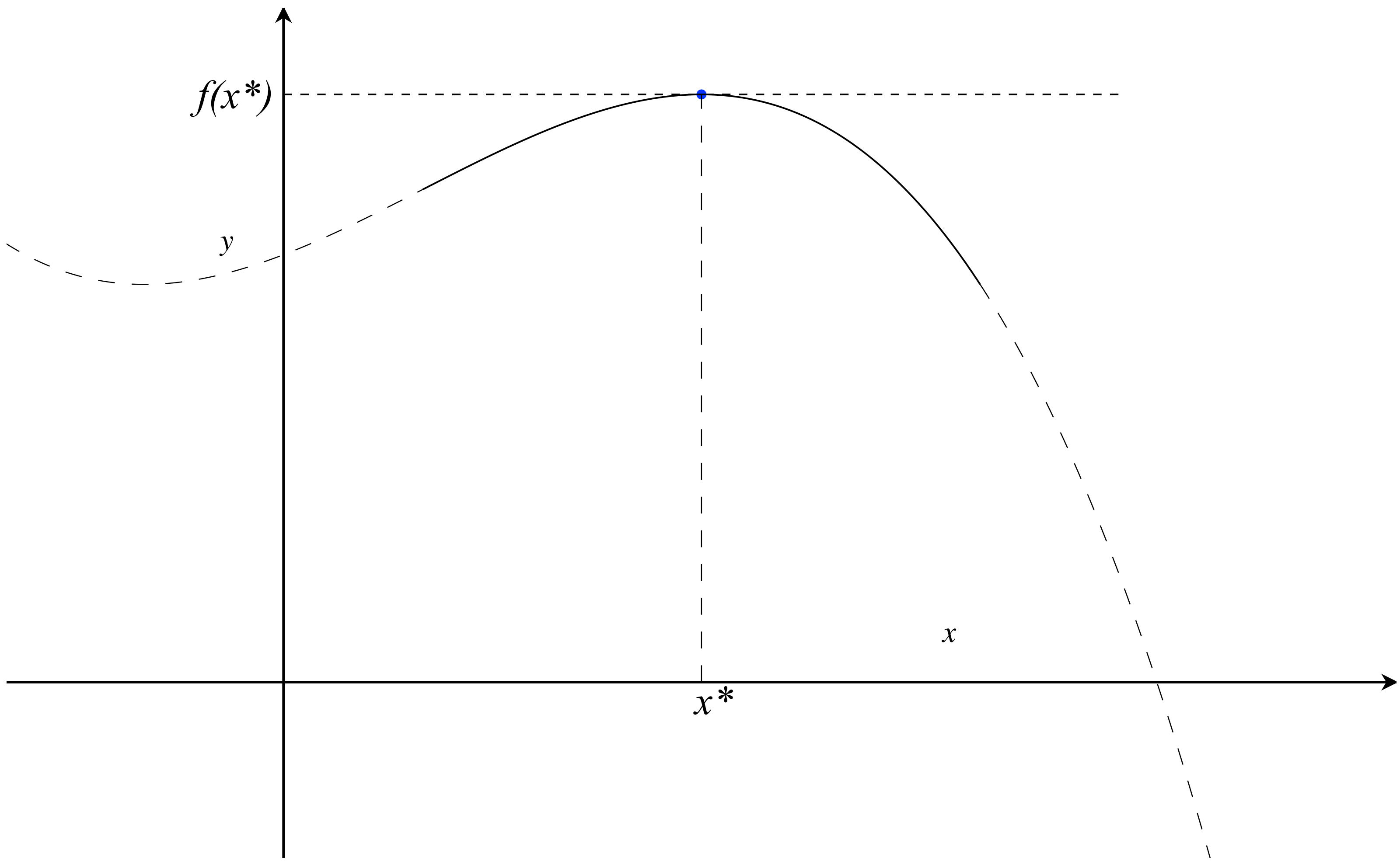
(*) If $f''(x^*) > 0$, then the graph of $y = f(x)$ is concave up around $(x^*, f(x^*))$, and if it is also true that $f'(x^*) = 0$, then the tangent line at $(x^*, f(x^*))$ is horizontal. The only way to draw this is with $f(x^*)$ being a relative minimum.

(*) Likewise, if $f''(x^*) < 0$, then the graph of $y = f(x)$ is concave down around $(x^*, f(x^*))$, and if it is also true that $f'(x^*) = 0$, then the tangent line at $(x^*, f(x^*))$ is horizontal. The only way to draw this is with $f(x^*)$ being a relative maximum.

(*) These comments are illustrated on the next two figures.



Relative minimum



Relative maximum