The product rule.

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 1. $\dfrac{d}{dx}\big((x^2+4x+5)(5x+3)\big) = \dots$

$$\begin{aligned}
&= \left(\frac{d}{dx}(x^2+4x+5)\right)(5x+3) + (x^2+4x+5)\left(\frac{d}{dx}(5x+3)\right) \\
&= (2x+4)(5x+3) + 5(x^2+4x+5) \\
&= 10x^2+26x+12+5x^2+20x+25 \\
&= 15x^2+46x+37
\end{aligned}$$

Check: $(x^2+4x+5)(5x+3) = 5x^3+23x^2+37x+15$, so

$\dfrac{d}{dx}\big((x^2+4x+5)(5x+3)\big) = \dfrac{d}{dx}\big(5x^3+23x^2+37x+15\big) = \dots \quad \checkmark$

Example 2. Find the derivative of $f(x) = \dfrac{3x^2 + 2x + 1}{x^2 + 2}$.

$$
\begin{aligned}
f'(x) &= \frac{\left(3x^2 + 2x + 1\right)' \cdot \left(x^2 + 2\right) - \left(3x^2 + 2x + 1\right) \cdot \left(x^2 + 2\right)'}{\left(x^2 + 2\right)^2} \\
&= \frac{(6x + 2) \cdot \left(x^2 + 2\right) - \left(3x^2 + 2x + 1\right) \cdot 2x}{\left(x^2 + 2\right)^2} \\
&= \frac{\left(6x^3 + 2x^2 + 12x + 4\right) - \left(6x^3 + 4x^2 + 2x\right)}{\left(x^2 + 2\right)^2} \\
&= \frac{-2x^2 + 10x + 4}{\left(x^2 + 2\right)^2}
\end{aligned}
$$

Example 3. Find the interval(s) where the slope of $s = \dfrac{3t}{t^2+1}$ is positive.

The slope of this graph is positive at the points t where $ds/dt > 0$, and...

$$\begin{aligned}\frac{ds}{dt} = \left(\frac{3t}{t^2+1}\right)' &= \frac{(3t)'(t^2+1) - 3t(t^2+1)'}{(t^2+1)^2} \\ &= \frac{3(t^2+1) - 3t \cdot 2t}{(t^2+1)^2} \\ &= \frac{3-3t^2}{(t^2+1)^2} = \frac{3(1-t^2)}{(t^2+1)^2}.\end{aligned}$$

Observation: $3 > 0$ and $(t^2+1)^2 > 0$ for all t.

Therefore $ds/dt > 0$ when $1 - t^2 > 0$, i.e., the slope is positive when $-1 < t < 1$.

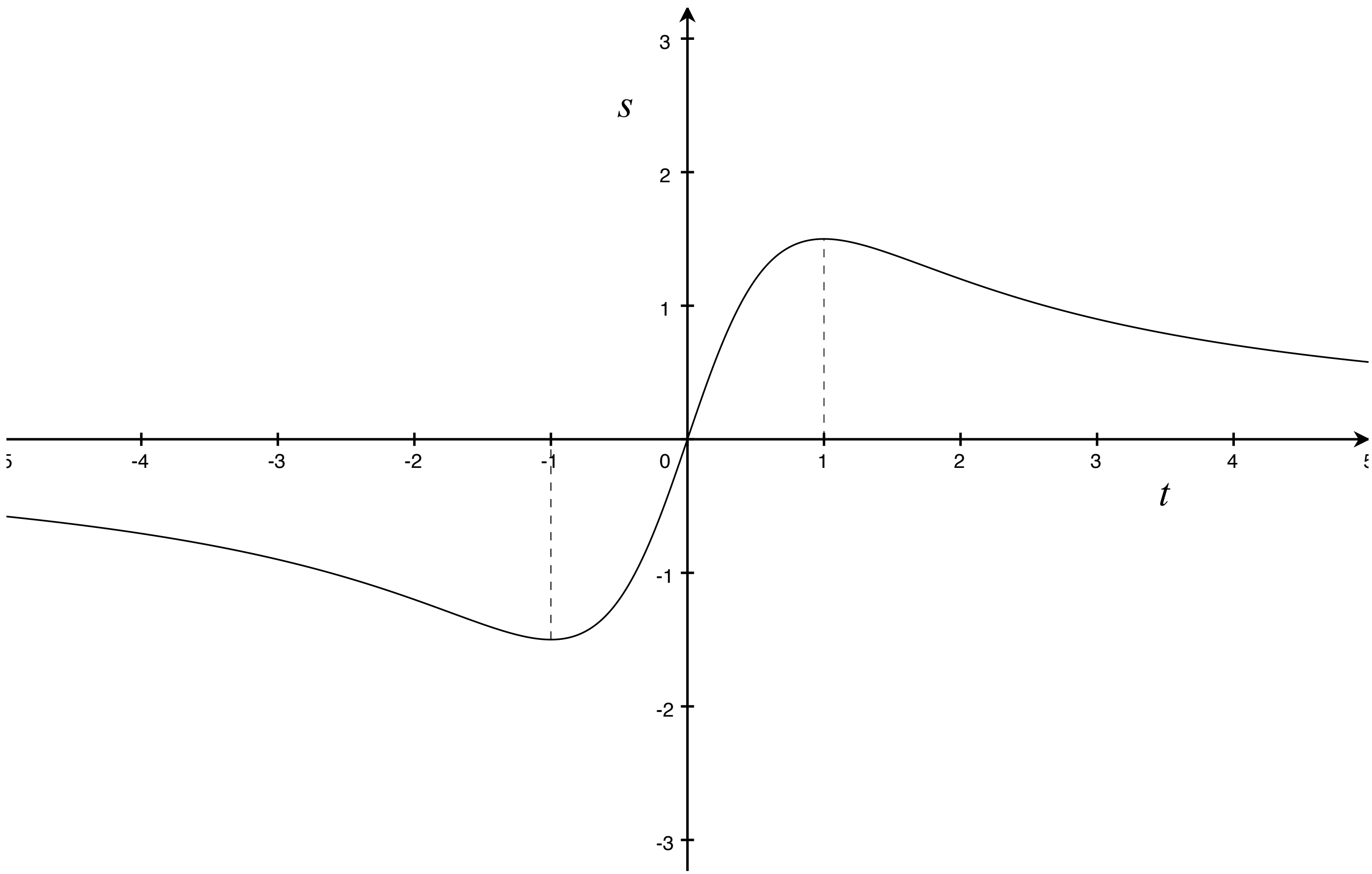

Figure 1: The graph of $s = \dfrac{3t}{t^2+1}$.

Example 4. Find the derivative of $g(x) = (2x+1)(x+3)(3x+5)$

$$\begin{aligned} g'(x) &= (2x+1)' \big[(x+3)(3x+5)\big] + (2x+1)\big[(x+3)(3x+5)\big]' \\ &= 2(x+3)(3x+5) + (2x+1)\big[(x+3)'(3x+5) + (x+3)(3x+5)'\big] \\ &= 2(x+3)(3x+5) + (2x+1)(3x+5) + 3(2x+1)(x+3) \\ &= (6x^2+28x+30) + (6x^2+13x+5) + (6x^2+27x+9) \\ &= 18x^2 + 68x + 44 \end{aligned}$$

More generally:

$$\frac{d}{dx}\big(f(x)g(x)h(x)\big) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

and

$$\begin{aligned} \frac{d}{dx}\big(f(x)g(x)h(x)j(x)\big) = {} & f'(x)g(x)h(x)j(x) + f(x)g'(x)h(x)j(x) \\ & + f(x)g(x)h'(x)j(x) + f(x)g(x)h(x)j'(x) \end{aligned}$$

etc.

Example 5. Find the derivative of $y = \dfrac{x^3 - 5x + 7}{2x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x^2-5)2x^2 - 4x(x^3-5x+7)}{(2x^2)^2} \\ &= \frac{6x^4 - 10x^2 - 4x^4 + 20x^2 - 28x}{4x^4} \\ &= \frac{2x^4 + 10x^2 - 28x}{4x^4} = \frac{x^4 + 5x^2 - 14x}{2x^4}\end{aligned}$$

Or *simplify before differentiating...*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x - \frac{5}{2}x^{-1} + \frac{7}{2}x^{-2}\right) = \frac{1}{2} + \frac{5}{2}x^{-2} - \frac{14}{2}x^{-3}$$

Make sure that you can see that the two answers are the same.

The product rule... Explanation:

$$
\begin{aligned}
(f(x)\cdot g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)\overbrace{-f(x)g(x+h) + f(x)g(x+h)}^{=0} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \\
&\qquad\qquad + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h\to 0} f(x)\cdot \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x)\cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

The quotient rule... Explanation:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)\overbrace{-f(x)g(x)+f(x)g(x)}^{=0}-f(x)g(x+h)}{g(x)g(x+h)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)-f(x)g(x)}{g(x)g(x+h)}\right)$$

$$- \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)g(x+h)-f(x)g(x)}{g(x)g(x+h)}\right)$$

$$
\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)-f(x)g(x)}{g(x)g(x+h)}\right) \\
&\qquad - \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)g(x+h)-f(x)g(x)}{g(x)g(x+h)}\right) \\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\left(\frac{f(x+h)g(x)-f(x)g(x)}{h}\right) \\
&\qquad - \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\left(\frac{f(x)g(x+h)-f(x)g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)g(x) \\
&\qquad - \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}f(x)\left(\frac{g(x+h)-g(x)}{h}\right) \\
&= \frac{1}{g(x)^2}\left(f'(x)g(x)-f(x)g'(x)\right) \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}
\end{aligned}
$$

Example. The consumption function of a small nation is given by

$$C = \frac{9Y^2 + 5Y + 100}{10Y + 1},$$

where both annual consumption C and annual income Y are measured in \$ billions.

1. Find the *marginal propensity to consume* and the *marginal propensity to save* when national income is \$8 billion.

Marginal propensity to consume: differentiate...

$$\frac{dC}{dY} = \frac{(18Y + 5)(10Y + 1) - 10(9Y^2 + 5Y + 100)}{(10Y + 1)^2}$$

$$= \frac{90Y^2 + 18Y - 995}{(10Y + 1)^2}$$

Then evaluate:

$$\left.\frac{dC}{dY}\right|_{Y=8} = \frac{90 \cdot 64 + 18 \cdot 8 - 995}{81^2} \approx 0.7402$$

Marginal propensity to save...?

Use the '*national accounting identity*':

$$C + S = Y \quad \Longrightarrow \quad \frac{dC}{dY} + \frac{dS}{dY} = 1 \quad \Longrightarrow \quad \frac{dS}{dY} = 1 - \frac{dC}{dY}$$

So:

$$\left.\frac{dS}{dY}\right|_{Y=8} = 1 - \left.\frac{dC}{dY}\right|_{Y=8} \approx 1 - 0.7402 = 0.2598$$

2. What happens to the MPC as income continues to grow? What does this say about national consumption when income is large?

$$\lim_{Y \to \infty} \frac{dC}{dY} = \lim_{Y \to \infty} \frac{90Y^2 + 18Y - 995}{(10Y + 1)^2} = \lim_{Y \to \infty} \frac{90Y^2 + 18Y - 995}{100Y^2 + 20Y + 1} = 0.9$$

Interpretation: *When income is large, the nation will tend to consume about* \$0.90 *of each additional dollar of income.*