Key observations:

1. If $f(x)$ is increasing in an interval (a, b) and f is differentiable there, then $f'(x) > 0$ in (a, b).

2. If $f(x)$ is decreasing in an interval (a, b) and f is differentiable there, then $f'(x) < 0$ in (a, b).

3. If $f(x)$ has a local minimum or maximum value at x^*, then the graph of $y = f(x)$ ***changes direction*** at x^*.

4. If $f(x)$ has a local minimum or maximum value at x^*, then $f'(x)$ ***changes sign*** at x^*.

Conclusion: If $f(x^*)$ is a relative extreme value, then...

(i) $f'(x^*) = 0$ *or*

(ii) $f'(x)$ *is not defined at* x^*.

Definition: If $f'(x^*) = 0$ or $f'(x)$ is not defined at x^*, then x^* is called a *critical point* of the function $f(x)$ and $f(x^*)$ is called a *critical value.*

All relative extreme values occur at critical points!

Example 1. Find the critical point(s) and critical value(s) of the function

$$f(x) = \frac{1}{3}x^3 - x^2 - 8x + 7.$$

First, differentiate:

$$f'(x) = x^2 - 2x - 8$$

Observe that $f'(x)$ is defined for all x, so we need only solve $f'(x) = 0$:

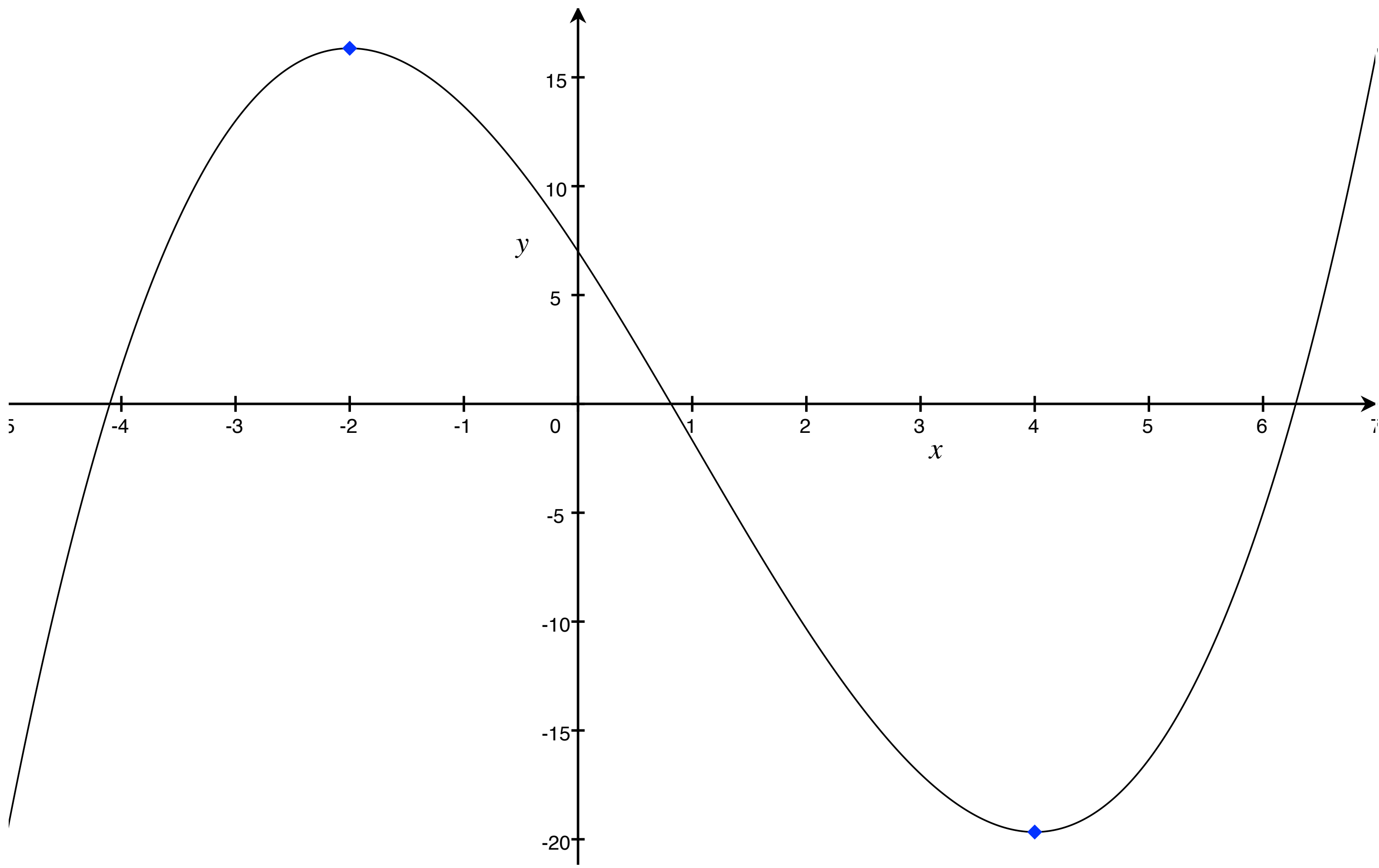
$$f'(x) = 0 \implies x^2 - 2x - 8 = 0 \implies x = \frac{2 \pm \sqrt{4 + 32}}{2}$$

There are two critical points, $x_1 = -2$ and $x_2 = 4$, with corresponding critical values

$$y_1 = f(-2) = \frac{49}{3} \quad \text{and} \quad y_2 = f(4) = -\frac{59}{3}.$$

Terminology: If x^* is a critical point of $f(x)$, then the point $(x^*, f(x^*))$ is called a critical point *on the graph* of $y = f(x)$.

Graph of $y = \frac{1}{3}x^3 - x^2 - 8x + 7$, with critical points (blue diamonds).



Example 2. Find the critical point(s) and critical value(s) of the function

$$w = 3t^{1/3}e^{-t^2/24}$$

Differentiate (and simplify!):

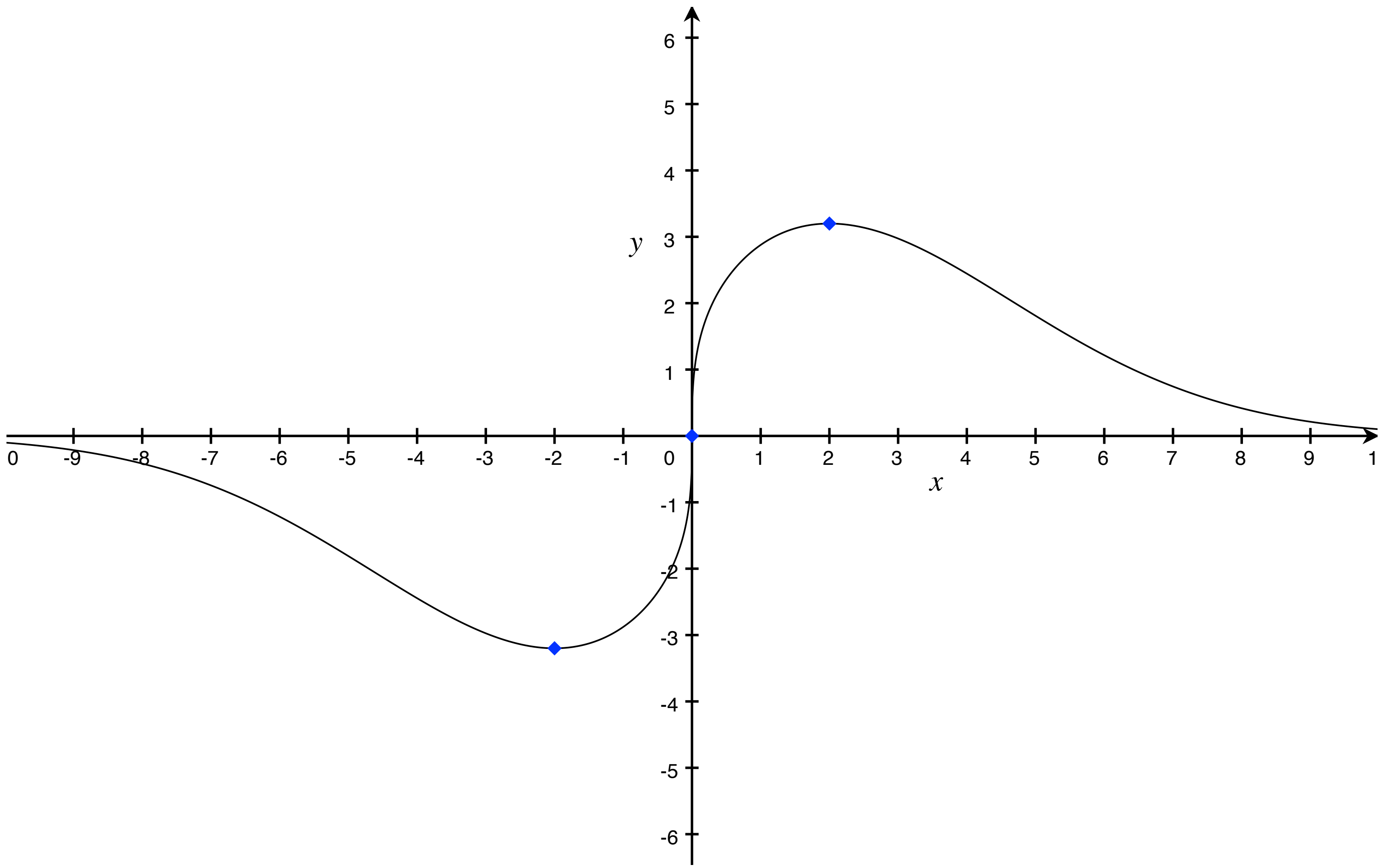
$$\begin{aligned}
\frac{dw}{dt} &= t^{-2/3}e^{-t^2/24} - \frac{1}{4}t^{4/3}e^{-t^2/24} && \text{product rule and chain rule}\\
&= e^{-t^2/24}\left(\frac{1}{t^{2/3}} - \frac{t^{4/3}}{4}\right) && \text{factor}\\
&= e^{-t^2/24}\left(\frac{4-t^2}{4t^{2/3}}\right) && \text{common denominator}
\end{aligned}$$

Analysis: dw/dt is undefined when $t = 0$ (because $4 \cdot 0^{2/3} = 0$) and $dw/dt = 0$ when $4 - t^2 = 0$ (because $e^u > 0$ for all real u).

Critical points: $t_1 = -2$, $t_2 = 0$ and $t_3 = 2$.

Critical values: $w_1 = 3(-2)^{1/3}e^{-1/6} \approx -3.1995$, $w_2 = 0$ and $w_3 = 3(2)^{1/3}e^{-1/6} \approx 3.1995$.

Graph of $w = 3t^{1/3}e^{-t^2/24}$, with critical points (blue diamonds).



Observation: The critical value $w(0) = 0$ is neither a local minimum value nor a local maximum value.

Question: Having found a critical point x^* of $f(x)$, how do we determine the *nature* of the critical value $f(x^*)$: local maximum, local minimum or neither?

Answer 1: We can use the *first derivative test.*

Answer 2: We can use the *second derivative test* (later).

The *first derivative test* is based on the observation that if $f'(x)$ changes sign at the critical point x^*, then $f(x^*)$ is a local extreme value:

- If $f'(x) > 0$ to the left of x^* (increasing to the left) and $f'(x) < 0$ to the right of x^* (decreasing to the right), then $f(x^*)$ is a local ***maximum*** value.

- If $f'(x) < 0$ to the left of x^* (decreasing to the left) and $f'(x) > 0$ to the right of x^* (increasing to the right), then $f(x^*)$ is a local ***minimum*** value.

But... If $f'(x)$ has the ***same sign on both sides of*** x^*, then $f(x^*)$ is ***neither*** a minimum ***nor*** a maximum value.

Question: How far to the left or right of the critical point(s) can we/should we check the sign of the derivative?

Answer: The derivative ***can only change sign at a critical point***. This means that between two consecutive critical points the sign of the derivative is constant (either $+$ or $-$).

$\Rightarrow$ To determine the sign of $f'(x)$ to the left and to the right of a critical point x^*, we can 'test' $f'(x)$ at any point between x^* and the next critical points to the left and right of x^*.

If there are no critical points to the left (right) of x^*, then we can test $f'(x)$ at any point to the left (right) of x^*.

Example 1. (continued) The critical points of the function

$$f(x) = \frac{1}{3}x^3 - x^2 - 8x + 7$$

are $x_1 = -2$ and $x_2 = 4$.

To ***classify the critical values***, $y_1 = f(-2) = 49/3$ and $y_2 = f(4) = -59/3$, we may sample $f'(x) = x^2 - 2x - 8$ at

- ... any point to the left of $x_1 = -2$;
- ... any point between x_1 and $x_2 = 4$;
- ... any point to the right of x_2.

So, for example, $f'(-3) = 7 > 0$ and $f'(0) = -8 < 0$, so $f(-2) = 49/3$ is a relative maximum value.

Likewise, $f'(0) < 0$ and $f'(5) = 7 > 0$, so $f(4) = -59/3$ is a relative minimum value.

Example 2. (continued) The critical points of

$$w = 3t^{1/3}e^{-t^2/24}$$

are $t_1 = -2$, $t_2 = 0$ and $t_3 = 2$. Also recall that $w' = e^{-t^2/24}\left(\frac{4-t^2}{4t^{2/3}}\right)$

First derivative test:

- $w'(-8) = -\frac{15}{4}e^{-8/3} < 0$ and $w'(-1) = \frac{3}{4}e^{-1/24} > 0$, and therefore $w(-2) \approx -3.1995$ is a local minimum value.
- $w'(-1) > 0$ and $w'(1) = \frac{3}{4}e^{-1/24} > 0$, so $w(0) = 0$ is neither a maximum nor a minimum value (the derivative did not change sign).
- $w'(1) > 0$ and $w'(8) = -\frac{15}{4}e^{-8/3} < 0$, so $w(2) \approx 3.1995$ is a local maximum value.

Example 3. Find the critical points of $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 1$ and *classify* the critical values (as local max, local min or neither).

Step 1. Differentiate: $f'(x) = 4x^3 - 12x^2 - 4x + 12$

Step 2. Solve $f'(x) = 0$...

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 - 4x + 12 = 4\left(x^3 - 3x^2 - x + 3\right) \\ &= 4\left(x^2(x-3) - (x-3)\right) = 4(x-3)(x^2-1) \\ &= 4(x-3)(x-1)(x+1) \end{aligned}$$

Critical points: $x_1 = -1$, $x_2 = 1$ and $x_3 = 3$.

Step 3. First derivative test.

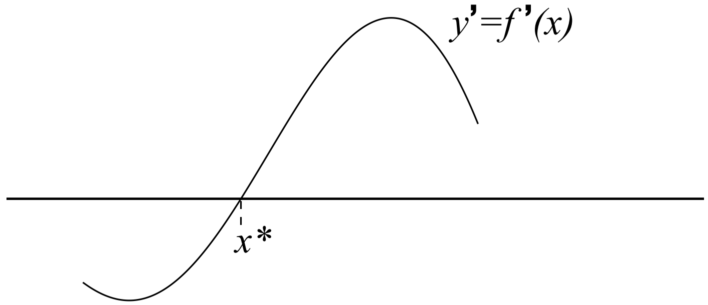
(*) $f'(-2) = -60 < 0$ and $f'(0) = 12 > 0$, so $f(-1) = -8$ is a local minimum value.

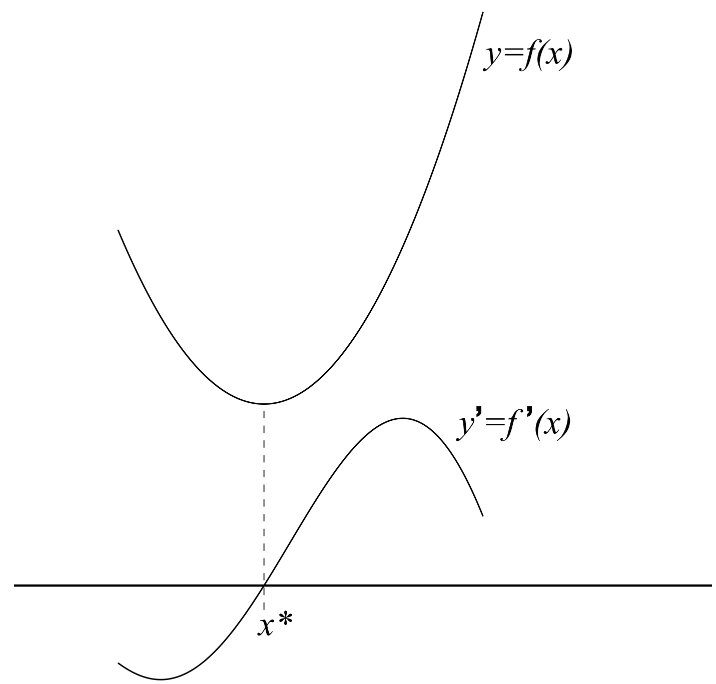
(*) $f'(0) > 0$ and $f'(2) = -12 < 0$, so $f(1) = 8$ is a local maximum value.

(*) $f'(2) < 0$ and $f'(4) = 60 > 0$, so $f(3) = -8$ is a local minimum value.

The second derivative test.

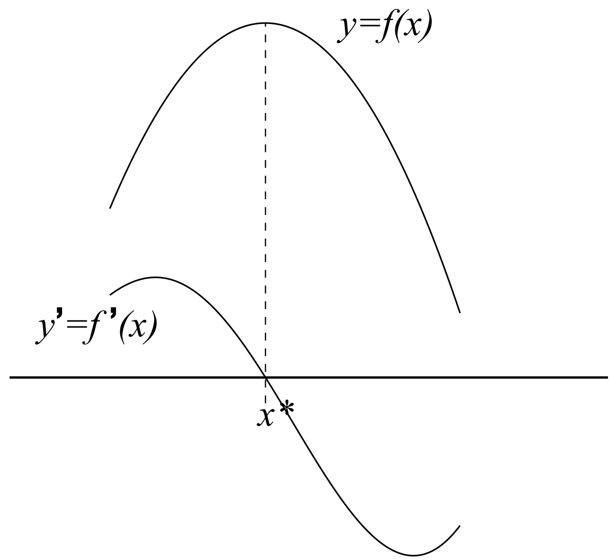
Suppose that $f'(x^*) = 0$ and $f''(x^*) > 0$. In this case $f'(x)$ is *increasing* around the point x^*. Now, the fact that $f'(x^*) = 0$ together with the fact that $f'(x)$ is increasing around x^* implies that $f'(x) < 0$ to the left of x^* and $f'(x) > 0$ to the right of x^*, as illustrated in the figure on the left below.





According to the *first derivative test*, this means that $f(x^*)$ is a local minimum value. This is illustrated in the figure on the right.

If $f'(x^*) = 0$ and $f''(x^*) < 0$, then $f'(x)$ is *decreasing* around x^* and therefore $f'(x) > 0$ to the left of x^* and $f'(x) < 0$ to the right of x^* (because $f'(x^*) = 0$). In this case, the first derivative test tells us that $f(x^*)$ must be a local maximum value, as illustrated below.



Summary: *The second derivative test*

(*) If $f'(x^*) = 0$ and $f''(x^*) > 0$, then $f(x^*)$ is a *local minimum value.*

(*) If $f'(x^*) = 0$ and $f''(x^*) < 0$, then $f(x^*)$ is a *local maximum value.*

(*) If $f'(x^*) = 0$ and $f''(x^*) = 0$, then *use the first derivative test.*