Differentiating logarithm functions.

To differentiate $y = \ln x$, we must return to the definition...

$$\frac{d}{dx}(\ln x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

and simplify, using algebraic properties of $\ln x$...

$$= \lim_{h \to 0} \frac{1}{h} \left(\ln(x+h) - \ln x\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h \to 0} \ln\left[\left(1 + \frac{h}{x}\right)^{1/h}\right]$$

and use the *continuity* of $\ln x$...

$$= \ln\left[\lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{1/h}\right]$$

What next?

Remember the special limit...

$$\lim_{u \to 0} (1+u)^{1/u} = e \ldots$$

do a little renaming...

$$\frac{h}{x} = u \implies h = ux \implies \frac{1}{h} = \frac{1}{ux} = \frac{1}{u} \cdot \frac{1}{x} \ldots$$

observe that $h \to 0$ *implies* $u \to 0$*...*

$$\begin{aligned}
\lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{1/h} &= \lim_{u \to 0} \, (1+u)^{\frac{1}{u} \cdot \frac{1}{x}} \\
&= \lim_{u \to 0} \left((1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}} \\
&= \left(\lim_{u \to 0} \, (1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}} \\
&\quad \text{(because } f(x) = a^{1/x} \text{ is continuous)} \\
&= e^{1/x}
\end{aligned}$$

Returning to $\dfrac{d}{dx}(\ln x)$...

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}\\ &\vdots\\ &= \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}\right]\\ &= \ln(e^{1/x})\\ &= \frac{1}{x}\ .\end{aligned}$$

I.e.,

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Example 1.

$$\frac{d}{dx}\left(3x^2 \ln x\right) = 6x \ln x + 3x^2 \cdot \frac{1}{x} = 6x \ln x + 3x$$

Example 2.

$$\frac{d}{dx} \ln\left(5x^2 + 3x + 1\right) = \frac{1}{5x^2 + 3x + 1} \cdot (10x + 3) = \frac{10x + 3}{5x^2 + 3x + 1}$$

Example 3. Differentiate $y = \ln\left(5x^2\right)$.

We can use the chain rule again:

$$y' = \frac{1}{5x^2} \cdot 10x = \frac{10x}{5x^2} = \frac{2}{x} \ldots$$

or we can simplify and then differentiate:

$$y = \ln\left(5x^2\right) = \ln 5 + \ln x^2 = \ln 5 + 2 \ln x$$

$$\implies y' = 0 + 2 \cdot \frac{1}{x} = \frac{2}{x}$$

Observation: If $b > 0$ and $b \neq 1$, then $\log_b x = \dfrac{\ln x}{\ln b}$, so

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b} \cdot \frac{1}{x}.$$

Example 4. Find the equation of the tangent line to the graph

$$y = \log_2\left(\frac{x^2 + 2x - 1}{4x - 3}\right)$$

at the point where $x = 1$.

(*) The line passes through the point $(1, y(1)) = (1, \log_2(2)) = (1, 1)$.

(*) The slope is $y'(1)$, and again we simplify before we differentiate:

$$y = \log_2\left(\frac{x^2 + 2x - 1}{4x - 3}\right) = \log_2(x^2 + 2x - 1) - \log_2(4x - 3)$$

$$\Longrightarrow \; y' = \frac{1}{\ln 2}\left(\frac{2x + 2}{x^2 + 2x - 1} - \frac{4}{4x - 3}\right) \;\Longrightarrow\; y'(1) = -\frac{2}{\ln 2}.$$

(*) The equation of the tangent line is

$$y = 1 - \frac{2}{\ln 2}(x - 1).$$

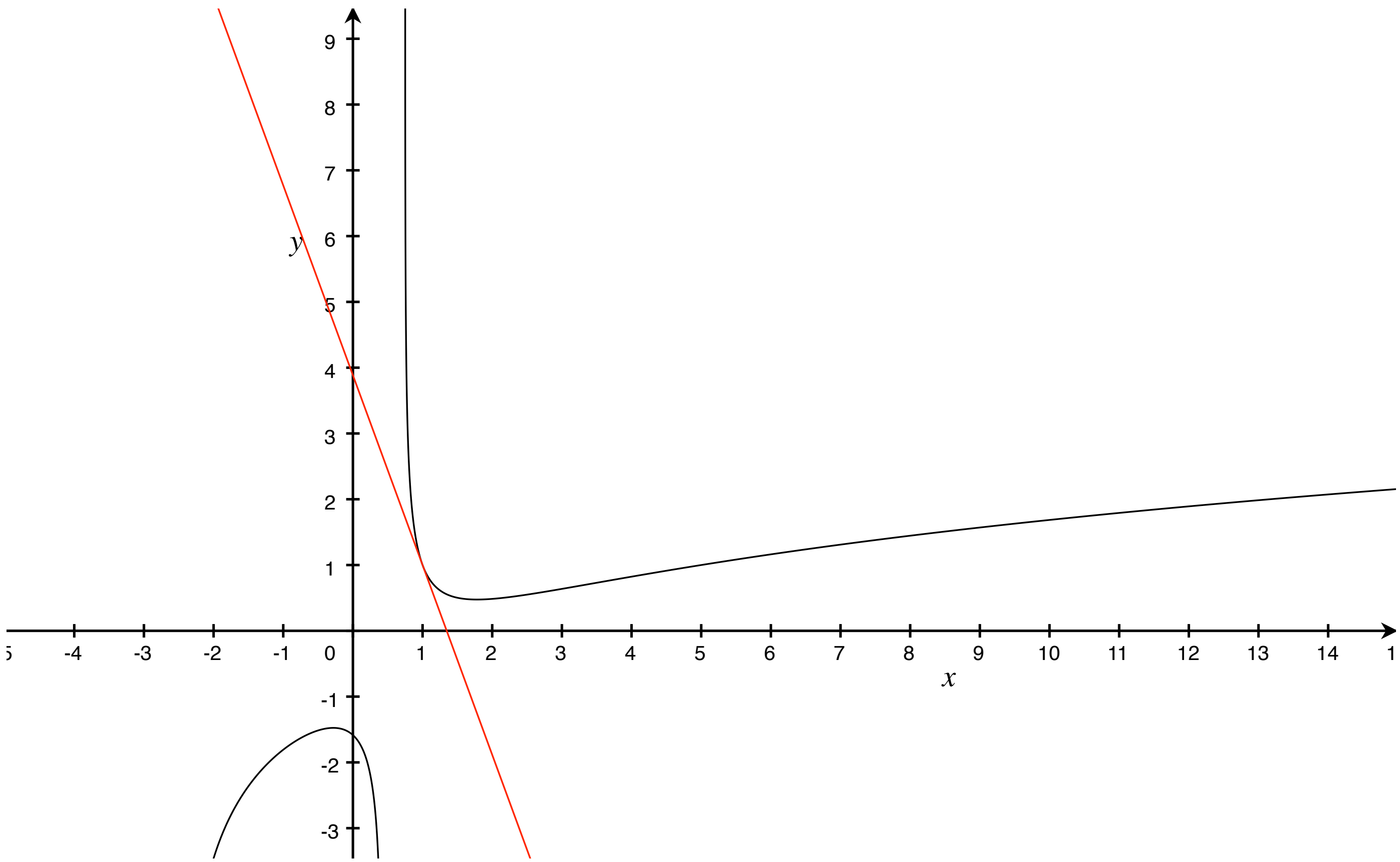

Figure 1: Graph of $y = \log_2\left(\frac{x^2+2x-1}{4x-3}\right)$ and its tangent line at $(1, 1)$

Logarithmic differentiation:

The chain rule tells us that

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

This is called the *logarithmic derivative* of $f(x)$.

(*) Sometimes $\frac{d}{dx}\ln(f(x))$ is easier to compute than $f'(x)$.

(*) In these cases we can use the identity

$$f'(x) = f(x) \cdot \left(\frac{d}{dx}\ln(f(x))\right)$$

to find $f'(x)$.

We will use this idea in one special, but **important** case...

Differentiating exponential functions.

Observation: $\ln(a^x) = x \ln a = (\ln a)x$ which is a simpler function (to differentiate) than a^x...

$$\frac{d}{dx} a^x = a^x \cdot \left(\frac{d}{dx} (\ln(a^x)) \right) = a^x \cdot \left(\frac{d}{dx} (\ln a) x \right) = a^x \cdot (\ln a) = (\ln a) a^x.$$

In particular, if $a = e$, then $\ln a = \ln e = 1$ and we have

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Example 5.

$$\frac{d}{dx} \left(3e^x x^{1/2} \right) = 3e^x x^{1/2} + 3e^x \cdot \frac{1}{2} x^{-1/2} = 3e^x \left(x^{1/2} + \frac{1}{2} x^{-1/2} \right)$$

Example 6.

$$\left(e^{x^2 - 3x + 1} \right)' = e^{x^2 - 3x + 1} (2x - 3)$$

Example 7. Find the marginal revenue function for the firm whose demand equation is given by

$$p = 15e^{2-0.05q}.$$

First, find the revenue function

$$r = pq = 15qe^{2-0.05q}.$$

Now differentiate, using the product and chain rules:

$$\frac{dr}{dq} = 15e^{2-0.05q} + 15qe^{2-0.05q}(-0.05) = (15 - 0.75q)e^{2-0.05q}.$$

The consumption function for a small country is given by

$$C = \ln\left(\frac{e^{0.95Y}}{e^{0.2Y}+5}\right),$$

where Y is national income, measured in \$billions.

(a) How much is consumed when $Y = 10$?

$$C(10) = \ln\left(\frac{e^{9.5}}{e^{2}+5}\right) \approx 6.983$$

(b) What is the marginal propensity to consume when $Y = 10$?

$$\begin{aligned}
\frac{dC}{dY} &= \frac{d}{dY}\left(\ln\left(\frac{e^{0.95Y}}{e^{0.2Y}+5}\right)\right) \\
&= \frac{d}{dY}\left(\ln\left(e^{0.95Y}\right) - \ln\left(e^{0.2Y}+5\right)\right) \\
&= \frac{d}{dY}\left(0.95Y - \ln\left(e^{0.2Y}+5\right)\right) = 0.95 - \frac{0.2e^{0.2Y}}{e^{0.2Y}+5}
\end{aligned}$$

$$\left.\frac{dC}{dY}\right|_{Y=10} = 0.95 - \frac{0.2e^{2}}{e^{2}+5} \approx 0.8307$$

(c) Compute the limit $\displaystyle\lim_{Y\to\infty}\frac{dC}{dY}$, and interpret the result.

$$\begin{aligned}
\lim_{Y\to\infty}\frac{dC}{dY} &= \lim_{Y\to\infty}\left(0.95-\frac{0.2e^{0.2Y}}{e^{0.2Y}+5}\right)\\
&= 0.95-\lim_{Y\to\infty}\frac{0.2e^{0.2Y}\cdot e^{-0.2Y}}{(e^{0.2Y}+5)\cdot e^{-0.2Y}}\\
&= 0.95-\lim_{Y\to\infty}\frac{0.2}{1+5e^{-0.2Y}} = 0.95-\frac{0.2}{1} = 0.75
\end{aligned}$$

Interpretation: *When income grows large, the nation will tend to consume \$0.75 of each additional dollar of income.*