Definition: If $y = f(x)$, then

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Rules and formulas:

1. If $f(x) = C$ (a constant function), then $f'(x) = 0$.

2. If $f(x) = x^k$ (a power function), then $f'(x) = kx^{k-1}$.

3. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.

4. $(C \cdot f(x))' = C \cdot f'(x)$

Notation:

$$\frac{d}{dx}\big(f(x)\big) = f'(x).$$

Example.

$$\begin{aligned}\frac{d}{dx}\left(\frac{3x^2-5x+7}{6x}\right) &= \frac{d}{dx}\left(\frac{3x^2}{6x}\right)-\frac{d}{dx}\left(\frac{5x}{6x}\right)+\frac{d}{dx}\left(\frac{7}{6x}\right)\\ &= \frac{d}{dx}\left(\frac{1}{2}x\right)-\frac{d}{dx}\left(\frac{5}{6}\right)+\frac{d}{dx}\left(\frac{7}{6}x^{-1}\right)\\ &= \frac{1}{2}-0+\frac{7}{6}\cdot\frac{d}{dx}\left(x^{-1}\right)\\ &= \frac{1}{2}+\frac{7}{6}(-1)x^{-2}=\frac{1}{2}-\frac{7}{6}x^{-2}\left(=\frac{3x^2-7}{6x^2}\right)\end{aligned}$$

Setting $f(x+\Delta x) - f(x) = \Delta y$, we can rewrite the definition of the derivative as

$$f'(x) = \lim_{h \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

(*) This leads to another common way of denoting the derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

(*) **Notation.** We use the following, interchangeable notations for the derivative of the function $y = f(x)$:

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}.$$

(*) **Evaluation notation.** If x_0 is a specific point and we want to evaluate $f'(x)$ at x_0, we write this in one of the following ways:

$$y'(x_0) = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0}.$$

Linear Approximation:

We begin with the definition of the derivative for a function $y = f(x)$ at a point x_0:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

" $\lim_{\Delta x \to 0}$ " in the equation above means that if $\Delta x \approx 0$ (but $\Delta x \neq 0$), then

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx f'(x_0). \tag{1}$$

Multiplying this approximate equality by Δx results in a very useful approximation:

$$f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x. \tag{2}$$

Observation: If $|\Delta x| < 1$, then the second approximation (2) is even more accurate than the first approximation (1).

Two useful variants:

1. If we write $\Delta y = f(x_0 + \Delta x) - f(x_0)$, then we can express the approximation (2) as

$$\Delta y \approx f'(x_0)\Delta x. \tag{3}$$

2. If we write $x = x_0 + \Delta x$, so $\Delta x = x - x_0$, and add $f(x_0)$ to both sides of (2), then this approximation takes the form

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{4}$$

Observations:

1. The function of x on the right of (4) is a *linear* function and this approximation formula says that if $x \approx x_0$, then we can approximate $f(x)$ by the linear function $T(x) = f(x_0) + f'(x_0)(x - x_0)$. For this reason, this approximation is frequently called ***linear approximation***.

2. This approximation is also called *tangent line approximation* because the graph of the linear function $y = T(x)$ above is the tangent line to the graph $y = f(x)$ at the point $(x_0, f(x_0))$.

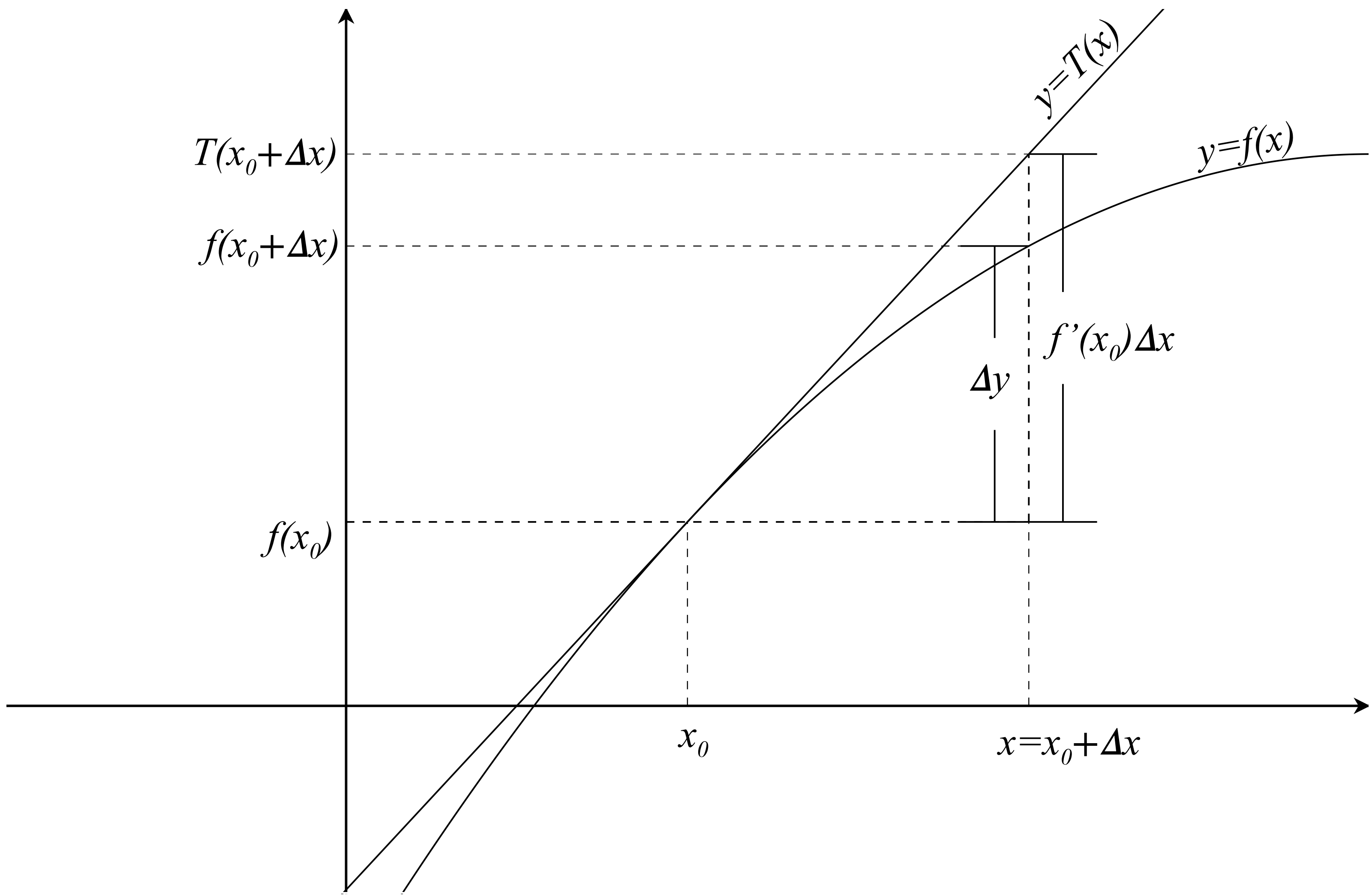

Figure 1: Linear approximation, illustrated.

Example 1. Find an approximate value for $\sqrt{26}$.

(*) Write $f(x) = \sqrt{x} = x^{1/2}$, then $f'(x) = \dfrac{1}{2}x^{-1/2}$.

(*) We don't know $\sqrt{26}$ but we do know that $\sqrt{25} = 5$, so we set $x_0 = 25$ and $x = 26$. Then, using linear approximation we have

$$\sqrt{26} = f(26) \approx f(25) + f'(25)(26 - 25) = 5 + \frac{1}{10} \cdot 1 = 5.1$$

(*) My calculator says that $\sqrt{26} = 5.09901951\ldots$. Based on this, the error of this approximation is less than 0.001.

Question: How can we obtain a more accurate estimate of $\sqrt{26}$ using linear approximation?

Answer: Choose x_0 closer to 26, so that $\Delta x = 26 - x_0$ is closer to 0. For this to be useful, we also need to know $\sqrt{x_0}$ *precisely*, which means that $\sqrt{x_0}$ must be a rational number.

For example, we can choose $x_0 = 5.1^2 = 26.01$ which satisfies both conditions. In this case,

$$f(x_0) = \sqrt{26.01} = 5.1 \quad \text{and} \quad f'(x_0) = \frac{1}{2\sqrt{26.01}} = \frac{1}{10.2} = \frac{5}{51}.$$

With this choice, linear approximation gives

$$\begin{aligned}\sqrt{26} = f(26) &\approx f(26.01) + f'(26.01)(26 - 26.01) \\ &= 5.1 + \frac{5}{51}\left(-\frac{1}{100}\right) \\ &= \frac{51}{10} - \frac{1}{1020} = \frac{5201}{1020} = 5.0990196078\ldots\end{aligned}$$

Comparing *this* estimate to the calculator value for $\sqrt{26}$ shows that the error of approximation is less than 0.0000001.

Example 2. The *marginal propensity to consume* of a small nation is given by

$$\frac{dC}{dY} = \frac{9Y + 10}{10Y + 1},$$

where the nation's income Y and consumption $C = f(Y)$ are both measured in billions of dollars.

The nation's current income is \$8 billion. By approximately how much will consumption increase if income increases by \$400 *million.*

First, observe that $\Delta Y = \dfrac{400,000,000}{1,000,000,000} = 0.4$, because of the units of measurement.

Now use linear approximation in the form of equation (3) (← click)

$$\Delta C \approx \left.\frac{dC}{dY}\right|_{Y=8} \cdot \Delta Y = \frac{9 \cdot 8 + 10}{10 \cdot 8 + 1} \cdot 0.4 \approx 0.405$$

Interpretation: Based on this model, if national income increases by \$400 million from its current level, national consumption will increase by about \$405 million (so the nation will incur about \$5 million in debt).

Economic terminology.

If $c = f(q)$ is a cost function (c is the cost of producing q units of output), then the cost of producing the next unit of output is called the *marginal cost*. I.e.,

$$\text{marginal cost} = MC = c(q+1) - c(q).$$

If $\dfrac{dc}{dq}$ is the derivative of the cost function, then linear approximation says that

$$MC = \Delta c \approx \left.\frac{dc}{dq}\right|_{q=q_0} \cdot \Delta q = \left.\frac{dc}{dq}\right|_{q=q_0}$$

because $\Delta q = 1$ in this case.

(*) We call $\dfrac{dc}{dq}$ the *marginal cost function*.

(*) More generally, if $v = g(u)$ is any function involving economic variables, the derivative dv/du is typically called the *marginal* _____ function, or the the *marginal* _____ of/to _____.