Properties of limits.

From the ***definition*** of the limit, the following *shortcuts* can be derived.

1. If $f(x) = c$ is a constant function, then $\lim\limits_{x \to a} f(x) = c$, for any a.

2. If n is a positive integer, then $\lim\limits_{x \to a} x^n = a^n$, for any a.

3. If β is any real number, then $\lim\limits_{x \to a} x^\beta = a^\beta$, for any $a > 0$.

4. If $a > 0$, then $\lim\limits_{x \to a} \ln x = \ln a$

5. For all real numbers a, $\lim\limits_{x \to a} e^x = e^a$

$\star$ If the limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ both exist, then ...

6. $$\lim_{x \to a} (f(x) \pm g(x)) = \left(\lim_{x \to a} f(x)\right) \pm \left(\lim_{x \to a} g(x)\right)$$

7. $$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$$

8. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided that } \lim_{x \to a} g(x) \neq 0.$$

Properties of limits (continued).

9. If $\lim_{x\to a} f(x) = L$ and $\lim_{u\to L} g(u) = M$, then $\lim_{x\to a} g(f(x)) = M$.

$\star$ This rule has some useful special cases: If $\lim_{x\to a} f(x) = L$, then...

9.1 If n is a positive integer, then $\lim_{x\to a} (f(x)^n) = \left(\lim_{x\to a} f(x)\right)^n = L^n$.

9.2 If $L > 0$, then $\lim_{x\to a} \left(f(x)^\beta\right) = \left(\lim_{x\to a} f(x)\right)^\beta = L^\beta$, for any β.

9.3 If $L > 0$, then $\lim_{x\to a} (\ln(f(x))) = \ln\left(\lim_{x\to a} f(x)\right) = \ln(L)$.

9.4 $\lim_{x\to a} \left(e^{f(x)}\right) = e^{\left(\lim_{x\to a} f(x)\right)} = e^L$.

$\star$ And finally, another useful tool for evaluating limits:

10. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

$\Rightarrow$ *Either both limits exist and are the same, or neither limit exists.*

Example 1. Find $\lim_{x\to 3} \dfrac{x^2+1}{x-1}$.

Solution:

$$\lim_{x\to 3} \frac{x^2+1}{x-1} = \frac{\lim_{x\to 3} x^2+1}{\lim_{x\to 3} x-1}$$

using the rule for quotients

$$= \frac{\lim_{x\to 3} x^2 + \lim_{x\to 3} 1}{\lim_{x\to 3} x - \lim_{x\to 3} 1}$$

using the rule for sums and differences

$$= \frac{9+1}{3-1} = 5$$

using the rules for powers and constants.

Example 2. Find $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

Solution: Observe that the rule for quotients doesn't apply here, because $\lim\limits_{x \to 3} x - 3 = 0$. For this limit, we have to use some algebra and rule 10 first, as follows.

First, if $x \neq 3$, then

$$\frac{x^2 - 9}{x - 3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = x + 3.$$

Therefore, by rule 10:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} x + 3 = 3 + 3 = 6.$$

A special limit

$$\lim_{u \to 0} (1+u)^{1/u} = e \approx 2.7182818283459.$$

We can evaluate the limit in Example 2 from Wednesday, using this limit and the following steps.

(1) $(1+0.02x)^{1/x} = (1+0.02x)^{\frac{1}{0.02x}\cdot(0.02)} = \left[(1+0.02x)^{1/(0.02x)}\right]^{0.02}$

(2) This means that

$$\begin{aligned}\lim_{x \to 0} (1+0.02x)^{1/x} &= \lim_{x \to 0} \left[(1+0.02x)^{1/(0.02x)}\right]^{0.02} \\ &= \left[\lim_{x \to 0} (1+0.02x)^{1/(0.02x)}\right]^{0.02}\end{aligned}$$

because of property 9.2, above.

(3) Finally, rename $0.02x = u$ and observe that if $x \to 0$, then $u \to 0$, so

$$\left[\lim_{x \to 0} (1+0.02x)^{1/(0.02x)}\right]^{0.02} = \left[\lim_{u \to 0} (1+u)^{1/u}\right]^{0.02} = e^{0.02} \approx 1.02020134.$$

One-sided limits

In some examples, we want to consider the behavior of a function $f(x)$ on either side of the limiting point a ***separately***.

Example: Suppose that

$$f(x) = \begin{cases} \sqrt{x} & : \quad x \geq 0 \\ \sqrt{x^2+1} & : \quad x < 0 \end{cases}$$

What can we say about $\lim_{x\to 0} f(x)$?

(*) If $x > 0$ and $x \to 0$, then $f(x) = \sqrt{x} \to 0$. We say in this case

the limit of $f(x)$ as x approaches 0 *from the right* is equal to 0

and write

$$\lim_{x\to 0^+} f(x) = 0$$

This is called a right-hand limit.

(*) If $x < 0$ and $x \to 0$, then $f(x) = \sqrt{x^2+1} \to \sqrt{1} = 1$. We say in this case

the limit of $f(x)$ as x approaches 0 *from the left* is equal to 1

and write

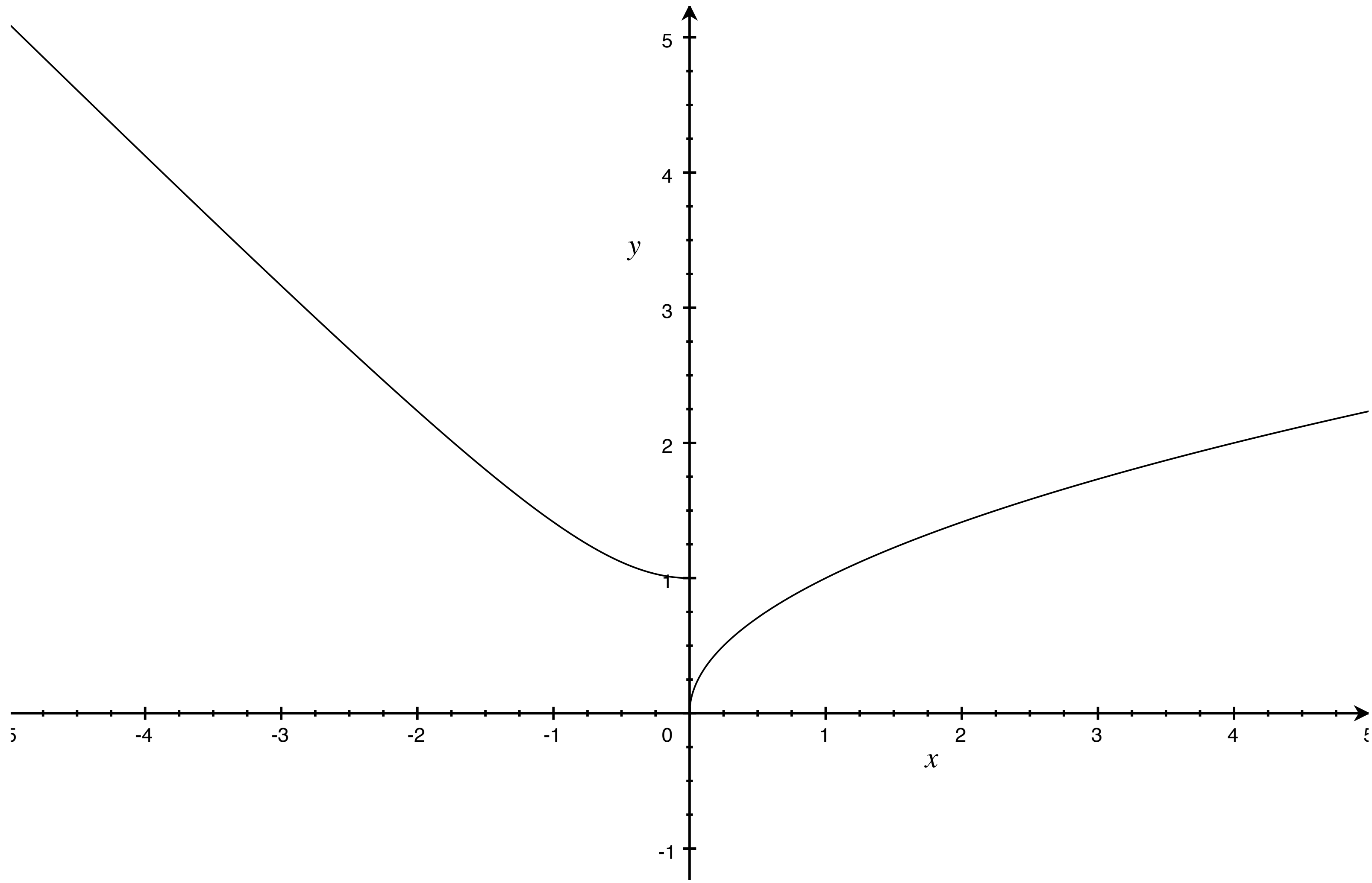
$$\lim_{x \to 0^-} f(x) = 1$$

This is called a left-hand limit.

(*) In this example both the left- and right-hand limits exist, but the *two-sided* limit

$$\lim_{x \to 0} f(x) \quad \textit{does not exist}$$

because there is no single number L that $f(x)$ approaches as $x \to 0$.



5
4
y
3
2
1
-4
-3
-2
-1
0
1
2
3
4
x
-1

Definitions:

The limit of $f(x)$ as x approaches a ***from the right*** is equal to L, written

$$\lim_{x \to a^+} f(x) = L,$$

if $|f(x) - L|$ can be made as small as we want by making $x - a$ as small as we need with $x > a$.

The limit of $f(x)$ as x approaches a ***from the left*** is equal to L, written

$$\lim_{x \to a^-} f(x) = L,$$

if $|f(x) - L|$ can be made as small as we want by making $a - x$ as small as we need with $x < a$.

Observation: $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist *and are equal to each other*. I.e.,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

Example: Suppose that

$$g(x) = \begin{cases} x^2 + 1 & : \quad x \geq 1 \\ \dfrac{x^2 - 1}{x - 1} & : \quad x < 1 \end{cases}$$

Find the limit $\lim_{x \to 1} g(x)$, or explain why it does not exist.

(*) $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} x^2 + 1 = 1 + 1 = 2.$

and

(*) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \dfrac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$

$= \lim_{x \to 1^-} x + 1 = 1 + 1 = 2$

Therefore

$$\lim_{x \to 1} g(x) = \lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) = 2$$

‘Infinite’ limits.

If $f(x)$ grows larger and larger without bound as x approaches some point a, then we say that $f(x)$ *is approaching infinity* as x approaches a and write

$$\lim_{x \to a} f(x) = \infty.$$

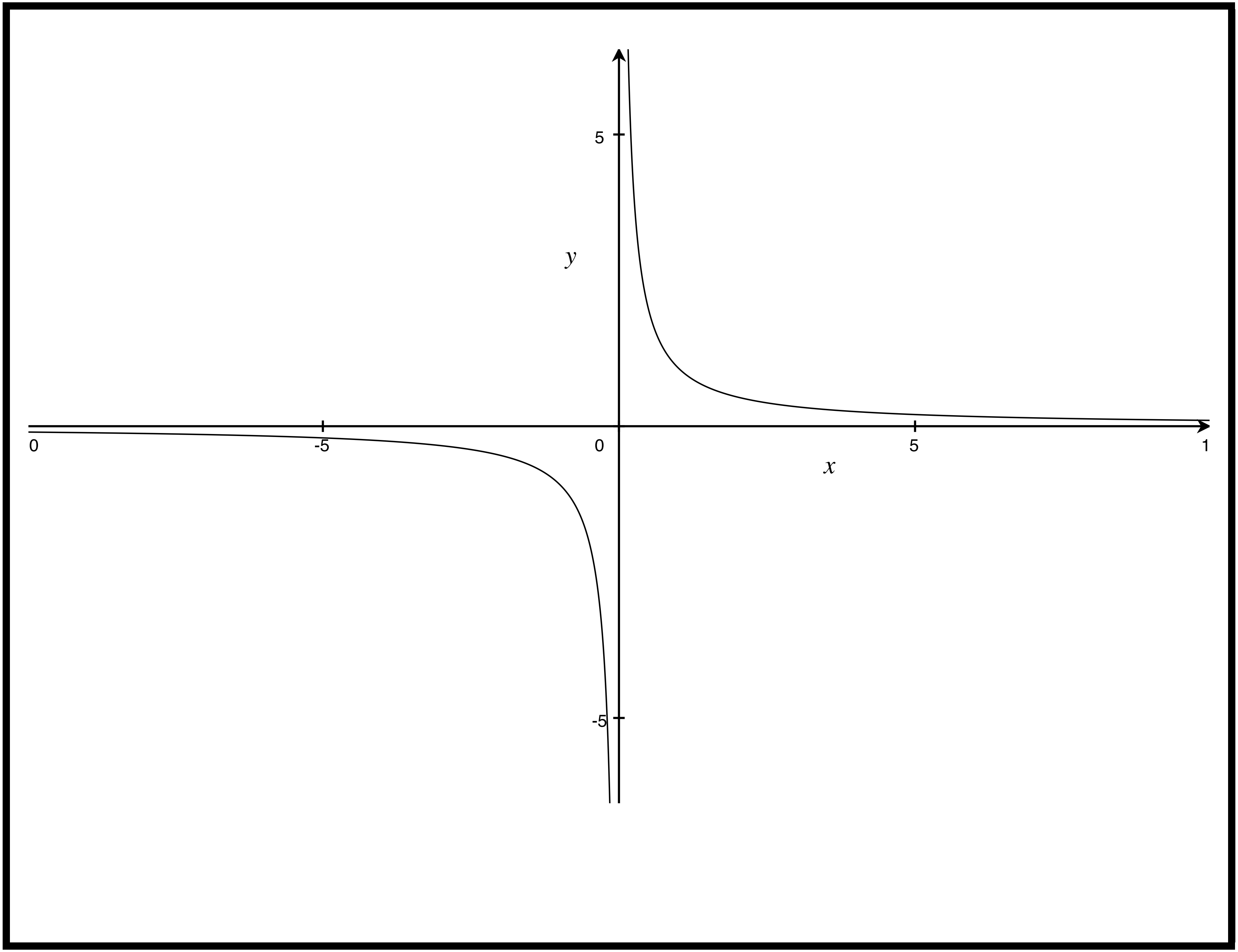
Likewise, If $f(x)$ grows more and more negative without (lower) bound as x approaches some point a, then we say that $f(x)$ *is approaching negative infinity* as x approaches a and write

$$\lim_{x \to a} f(x) = -\infty.$$

(*) If $f(x)$ exhibits one of these behaviors on only one side of a or the other, then we use one-sided limits to describe the situation.

Example: $\displaystyle\lim_{x \to 0} \frac{1}{x}$ does not exist, but we can observe that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$


5
y
-5
0
-5
0
5
1
x

Limits 'at infinity'

(*) The *limit at infinity* of a function $f(x)$ describes (certain aspects of) the behavior of $f(x)$ as the variable x grows larger and larger.

Example 1: What happens to the values of $f(x) = \dfrac{1}{x}$ as x grows without bound?

x	$\frac{1}{x}$
1	1
100	0.01
$10,000$	0.0001
$1,000,000$	0.000001
10^{100}	$0.\overbrace{00\ldots0}^{99\text{ 0s}}1$

Observations:

(i) As k grows bigger (and 10^k grows even faster), $\frac{1}{10^k}$ approaches 0.

(ii) If $10^k < x$, then $0 < \frac{1}{x} < \frac{1}{10^k}$, so $\frac{1}{x}$ approaches 0 as x grows large.

Definition:

The limit of $f(x)$ as x 'approaches infinity' is equal to L, written

$$\lim_{x\to\infty} f(x) = L,$$

if $|f(x) - L|$ can be made as small as we want by choosing x as large as necessary.

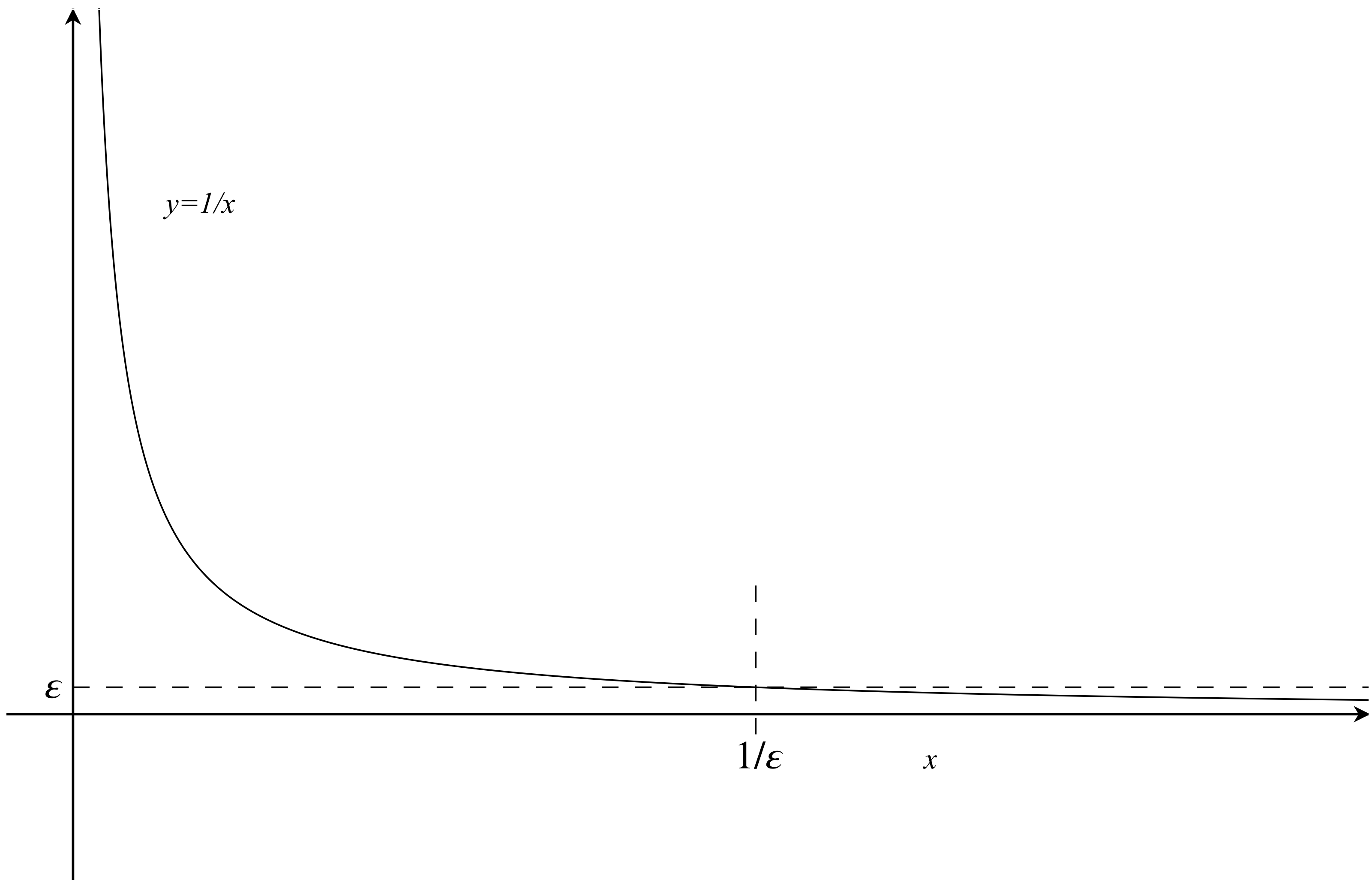
In other words, 'approaching infinity' means getting large without bound.

More formal definition:

$\lim_{x\to\infty} f(x) = L$ means that given $\varepsilon > 0$, there is an M such that $|f(x) - L| < \varepsilon$ for all $x > M$.

Example: $\lim_{x\to\infty} \frac{1}{x} = 0$ because given $\varepsilon > 0$, if $x > M = \frac{1}{\varepsilon}$, then

$$\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{M} = \frac{1}{1/\varepsilon} = \varepsilon.$$


y=1/x
ε
1/ε
x

1. The rules for constant functions and for sums and differences of limits are valid for limits at infinity.

2. The rules for products and quotients of limits are equally valid for limits at infinity, *as long as the component limits are finite.* In particular, the expressions $\dfrac{\infty}{\infty}$ and $0 \cdot \infty$ are meaningless.

3. If $k > 0$, then $\lim\limits_{x\to\infty} x^k = \infty$, i.e., x^k grows larger as x grows larger.

4. If $k < 0$, then $\lim\limits_{x\to\infty} x^k = \lim\limits_{x\to\infty} \dfrac{1}{x^{|k|}} = 0$.

5. More generally, if $\lim\limits_{x\to\infty} f(x) = \infty$, then $\lim\limits_{x\to\infty} \dfrac{1}{f(x)} = 0$.

6. If $a > 0$, then $\lim\limits_{x\to\infty} e^{ax} = \infty$ and therefore $\lim\limits_{x\to\infty} e^{-ax} = \lim\limits_{x\to\infty} \dfrac{1}{e^{ax}} = 0$.

7. *Exponential growth.* If $a > 0$, then for any k

$$\lim_{x\to\infty} \frac{x^k}{e^{ax}} = 0.$$