Example 1. What happens to the expression

$$P(x) = (1 + 0.02x)^{1/x}$$

as x approaches 1?

x	$P(x)$
1.5	~ 1.019901
1.25	~ 1.01995
1.1	~ 1.01998
1.01	~ 1.09998
0.5	1.0201
0.9	~ 1.02002
0.99	~ 1.020002

Observation:

As x approaches 1, $P(x)$ appears to be approaching $1.02 = P(1)$.

Example 2. What happens to the expression

$$P(x) = (1 + 0.02x)^{1/x}$$

as x approaches 0?

x	$P(x)$
1	1.02
$1/2$	1.0201
$1/12$	~ 1.020184356
$1/52$	~ 1.020197417
$1/365$	~ 1.020200781
$1/20000$	~ 1.020201329

The variable x can also approach 0 through negative values:

x	$P(x)$
-1	~ 1.020408163
$-1/2$	~ 1.020304051
$-1/10$	~ 1.020221772
$-1/80$	~ 1.020203891
$-1/200$	~ 1.020202360
$-1/10000$	~ 1.020201316

Observations:

(i) $P(x)$ is not defined when $x = 0$ (why?).

(ii) It appears that as x approaches 0, $P(x)$ approaches some number

$$\tau \approx 1.0202013\ldots.$$

Example 3. What happens to the expression

$$Q(x) = \frac{\sqrt{x} - 2}{x - 4}$$

as x approaches 0?

x	$Q(x)$
1	$1/3 \approx 0.3333$
$1/2$	~ 0.3694
$1/4$	0.4
$1/9$	$3/7 \approx 0.4286$
$1/100$	$10/21 \approx 0.4762$
$1/10000$	$19900/39999 \approx 0.4975$

Observations:

(i) As x approaches 0, $Q(x)$ appears to be approaching $0.5 = Q(0)$.

(ii) We can only approach 0 from the right in this case (why?).

Example 4. What happens to the expression

$$Q(x) = \frac{\sqrt{x} - 2}{x - 4}$$

as x approaches 4?

x	$Q(x)$
3	~ 0.26795
3.5	~ 0.25834
3.9	~ 0.25158
4.5	~ 0.24264
4.1	~ 0.24846
4.01	~ 0.24984

Observations:

(i) $Q(x)$ is not defined at $x = 4$ (why?).

(ii) As x approaches 4, $Q(x)$ appears to be approaching 0.25.

Definition of 'Limit':

The limit of $f(x)$ as x approaches a is equal to L, written

$$\lim_{x \to a} f(x) = L,$$

if we can make the difference $|f(x) - L|$ as small as we like by taking x sufficiently close to a, but ***not equal*** to a.

If there is ***no number L*** satisfying this condition, then the limit ***does not exist***.

The technical definition:

$$\lim_{x \to a} f(x) = L$$

if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

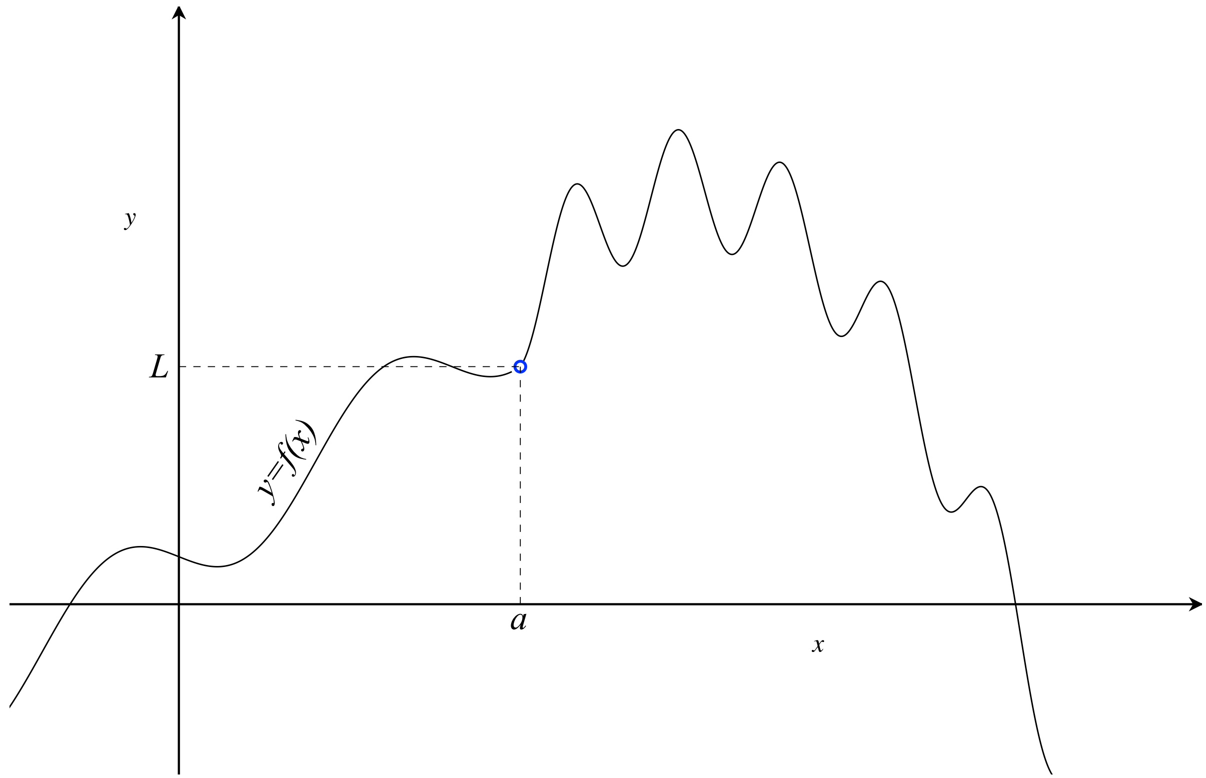

Figure 1: Graphic interpretation of $\lim_{x \to a} f(x) = L$

Notice that the value of the function $f(x)$ at the point $x = a$ does not appear in the graph above. The value $f(a)$ (if it exists) is irrelevant to the ***definition*** of the limit.

On the other hand in many cases, $\lim_{x \to a} f(x) = f(a)$, as we shall see.

Properties of limits.

From the ***definition*** of the limit, the following *shortcuts* can be derived.

1. If $f(x) = c$ is a constant function, then $\lim_{x\to a} f(x) = c$, for any a.

2. If n is a positive integer, then $\lim_{x\to a} x^n = a^n$, for any a.

3. If β is any real number, then $\lim_{x\to a} x^\beta = a^\beta$, for any $a > 0$.

4. If $a > 0$, then $\lim_{x\to a} \ln x = \ln a$

5. For all real numbers a, $\lim_{x\to a} e^x = e^a$

$\star$ If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then ...

6. $$\lim_{x\to a} (f(x) \pm g(x)) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right)$$

7. $$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

8. $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$ provided that $\lim_{x\to a} g(x) \neq 0$.

Properties of limits (continued).

9. If $\lim_{x\to a} f(x) = L$ and $\lim_{u\to L} g(u) = M$, then $\lim_{x\to a} g(f(x)) = M$.

$\star$ This rule has some useful special cases: If $\lim_{x\to a} f(x) = L$, then...

9.1 If n is a positive integer, then $\lim_{x\to a} (f(x)^n) = \left(\lim_{x\to a} f(x)\right)^n = L^n$.

9.2 If $L > 0$, then $\lim_{x\to a} \left(f(x)^\beta\right) = \left(\lim_{x\to a} f(x)\right)^\beta = L^\beta$, for any β.

9.3 If $L > 0$, then $\lim_{x\to a} (\ln(f(x))) = \ln\left(\lim_{x\to a} f(x)\right) = \ln(L)$.

9.4 $\lim_{x\to a} \left(e^{f(x)}\right) = e^{\left(\lim_{x\to a} f(x)\right)} = e^L$.

$\star$ And finally, another useful tool for evaluating limits:

10. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

$\Rightarrow$ *Either both limits exist and are the same, or neither limit exists.*

Example 4. (revisited)

We can use Property 10 (and some of the other properties) to evaluate the limit in Example 4 (and confirm our observation there).

First we observe that as long as $x \neq 4$ (and $x \geq 0$)

$$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{x-4} \cdot 1 = \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}.$$

We now proceed to evaluate the limit:

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} && \text{(property 10)} \\
&= \frac{\lim_{x\to 4} 1}{\lim_{x\to 4}(x^{1/2}+2)} && \text{(property 8 and } \sqrt{x} = x^{1/2}\text{)} \\
&= \frac{1}{\lim_{x\to 4} x^{1/2} + \lim_{x\to 4} 2} && \text{(properties 1 and 6)} \\
&= \frac{1}{4^{1/2}+2} = \frac{1}{4} && \text{(properties 1 and 3)}
\end{aligned}$$