Limits at infinity of *rational functions*.

Example. Find $\lim_{x\to\infty} \dfrac{2x^2+10x+100}{x^3+x^2+1}$.

(*) The rule for quotients doesn't work here, because

$$\lim_{x\to\infty} 2x^2+10x+100 = \lim_{x\to\infty} x^3+x^2+1 = \infty.$$

However, the denominator is *growing faster* than the numerator (why?), which indicates that the limit is probably 0.

To see that this is true, we use a little algebra to simplify:

$$\frac{2x^2+10x+100}{x^3+x^2+1} = \frac{\cancel{x^2}\left(2+\frac{10}{x}+\frac{100}{x^2}\right)}{x^{\cancel{3}}\left(1+\frac{1}{x}+\frac{1}{x^3}\right)} = \frac{1}{x}\cdot\frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}}$$

Therefore...

$$
\begin{aligned}
\lim_{x\to\infty} \frac{2x^2+10x+100}{x^3+x^2+1} &= \lim_{x\to\infty} \left(\frac{x^2}{x^3} \cdot \frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}} \right) \\
&= \lim_{x\to\infty} \left(\frac{1}{x} \cdot \frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}} \right) \\
&= \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}} \\
&= 0 \cdot \frac{\lim_{x\to\infty} 2+\dfrac{10}{x}+\dfrac{100}{x^2}}{\lim_{x\to\infty} 1+\dfrac{1}{x}+\dfrac{1}{x^3}} \\
&= 0 \cdot \frac{2+0+0}{1+0+0} = 0 \cdot 2 = 0.
\end{aligned}
$$

Observation: If U is very large and $a > 0$, then aU is very large. On the other hand, if $a < 0$, then aU is very negative, i.e., $aU < 0$ and $|aU|$ is very large.

Conclusions: If $\lim_{x \to \infty} f(x) = \infty$, then

$$\lim_{x \to \infty} af(x) = \infty$$

if $a > 0$ and

$$\lim_{x \to \infty} af(x) = -\infty$$

if $a < 0$.

Moreover, if $\lim_{x \to \infty} g(x) = a > 0$, then

$$\lim_{x \to \infty} g(x)f(x) = \infty$$

and if $\lim_{x \to \infty} g(x) = b < 0$, then

$$\lim_{x \to \infty} g(x)f(x) = -\infty$$

Example: Find $\lim_{x\to\infty} \dfrac{x^4+10x-5}{2x^3+x^2+1}$.

Simplify as before:

$$\frac{x^4+10x-5}{2x^3+x^2+1} = \frac{\cancel{x^4}\left(1+\frac{10}{x^3}-\frac{5}{x^4}\right)}{\cancel{x^3}\left(2+\frac{1}{x}+\frac{1}{x^3}\right)} = x \cdot \frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}}$$

Therefore

$$\lim_{x\to\infty} \frac{x^4+10x-5}{2x^3+x^2+1} = \lim_{x\to\infty} \left(x \cdot \frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}} \right)$$

Now, $\lim_{x\to\infty} x = \infty$ and

$$\lim_{x\to\infty} \frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}} = \frac{\lim_{x\to\infty} 1+\frac{10}{x^3}-\frac{5}{x^4}}{\lim_{x\to\infty} 2+\frac{1}{x}+\frac{1}{x^3}} = \frac{1+0-0}{2+0+0} = \frac{1}{2} > 0$$

so it follows that

$$\lim_{x\to\infty} \left(x \cdot \frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}} \right) = \infty.$$

Generalizing the arithmetic:

$$\frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0} = \frac{x^m \left(a_m + a_{m-1}x^{-1} + \cdots + a_0 x^{-m}\right)}{x^n \left(b_n + b_{n-1}x^{-1} + \cdots + b_0 x^{-n}\right)}$$

$$= x^{m-n} \cdot \frac{a_m + a_{m-1}x^{-1} + \cdots + a_0 x^{-m}}{b_n + b_{n-1}x^{-1} + \cdots + b_0 x^{-n}}$$

Observation:

$$\lim_{x\to\infty} \frac{a_m + a_{m-1}x^{-1} + \cdots + a_0 x^{-m}}{b_n + b_{n-1}x^{-1} + \cdots + b_0 x^{-n}} = \frac{\lim_{x\to\infty} a_m + a_{m-1}x^{-1} + \cdots + a_0 x^{-m}}{\lim_{x\to\infty} b_n + b_{n-1}x^{-1} + \cdots + b_0 x^{-n}}$$

$$= \frac{a_m + 0 + \cdots + 0}{b_n + 0 + \cdots + 0} = \frac{a_m}{b_n}$$

This allows us to formulate a rule for the limit at infinity of rational functions:

$$\lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0} = \begin{cases} 0 & : \quad m < n \\ a_n/b_n & : \quad m = n \\ \pm\infty & : \quad m > n \end{cases}$$

Continuity and continuous functions

Definitions:

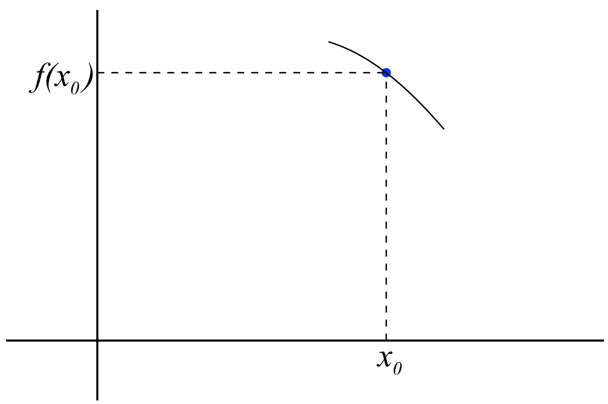
A. *The function $f(x)$ is* ***continuous at the point $x = x_0$*** *if*

(i) $f(x)$ is defined at $x = x_0$, and

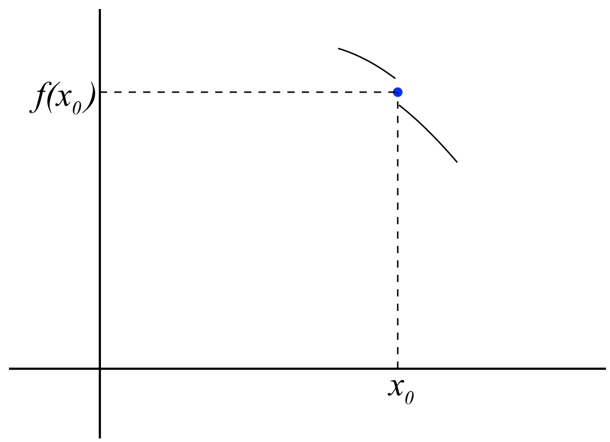
(ii) $\lim_{x \to x_0} f(x) = f(x_0)$.

(*) If $f(x)$ is ***not*** continuous at $x = x_0$, then we say that x_0 is *a point of discontinuity* of the function.

(*) Points of discontinuity leads to 'breaks' in the graph $y = f(x)$:



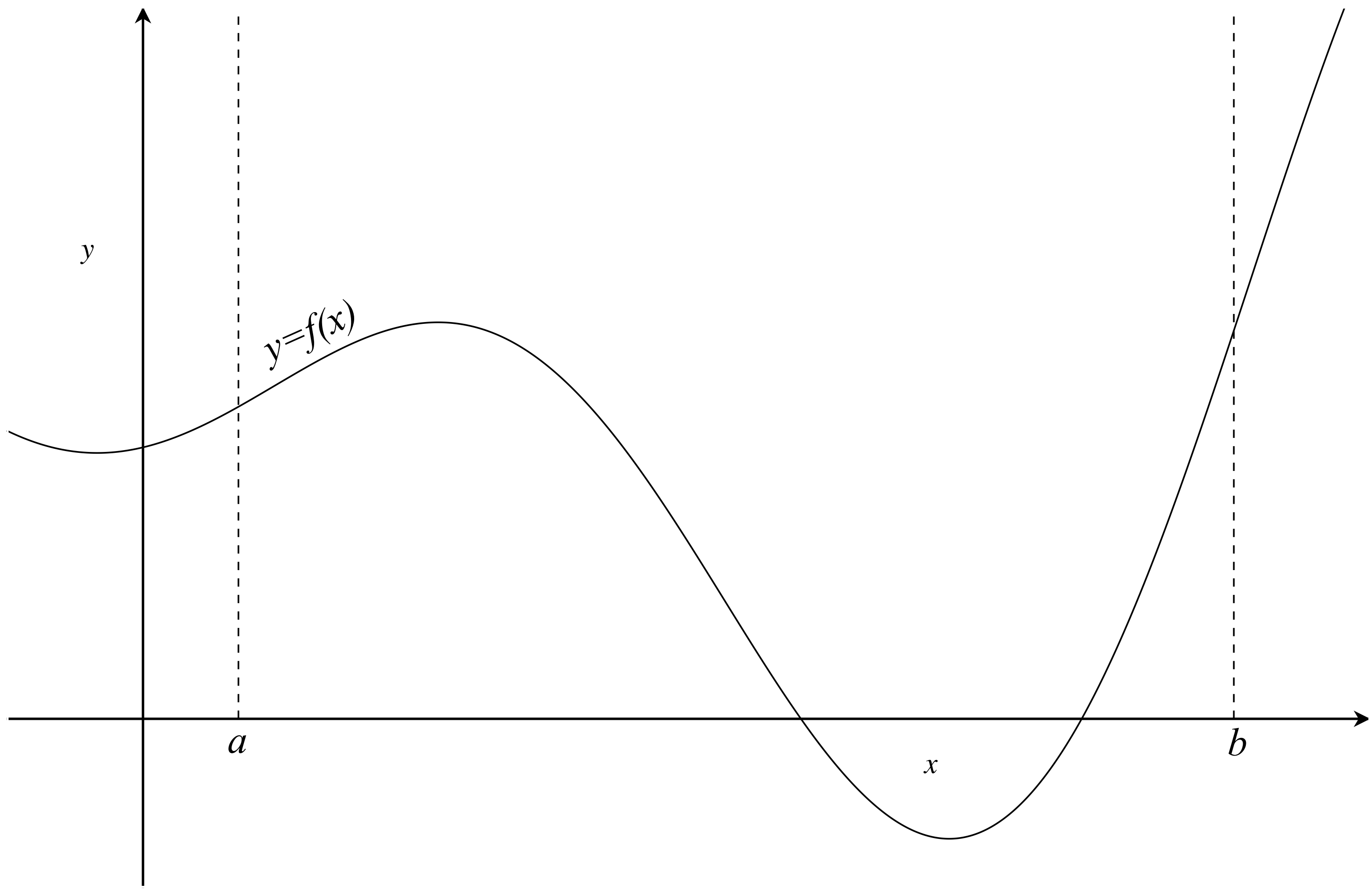
Continuous at x_0



Not continuous at x_0

B. *The function $f(x)$ is **continuous in the interval I=(a,b)** if $f(x)$ is continuous at every point x_0 in I.*

(*) This means that the graph $y = f(x)$ can be drawn with *one continuous stroke* over the interval (a, b).



Example: Consider the function $f(x) = \dfrac{x^2+1}{x+3}$.

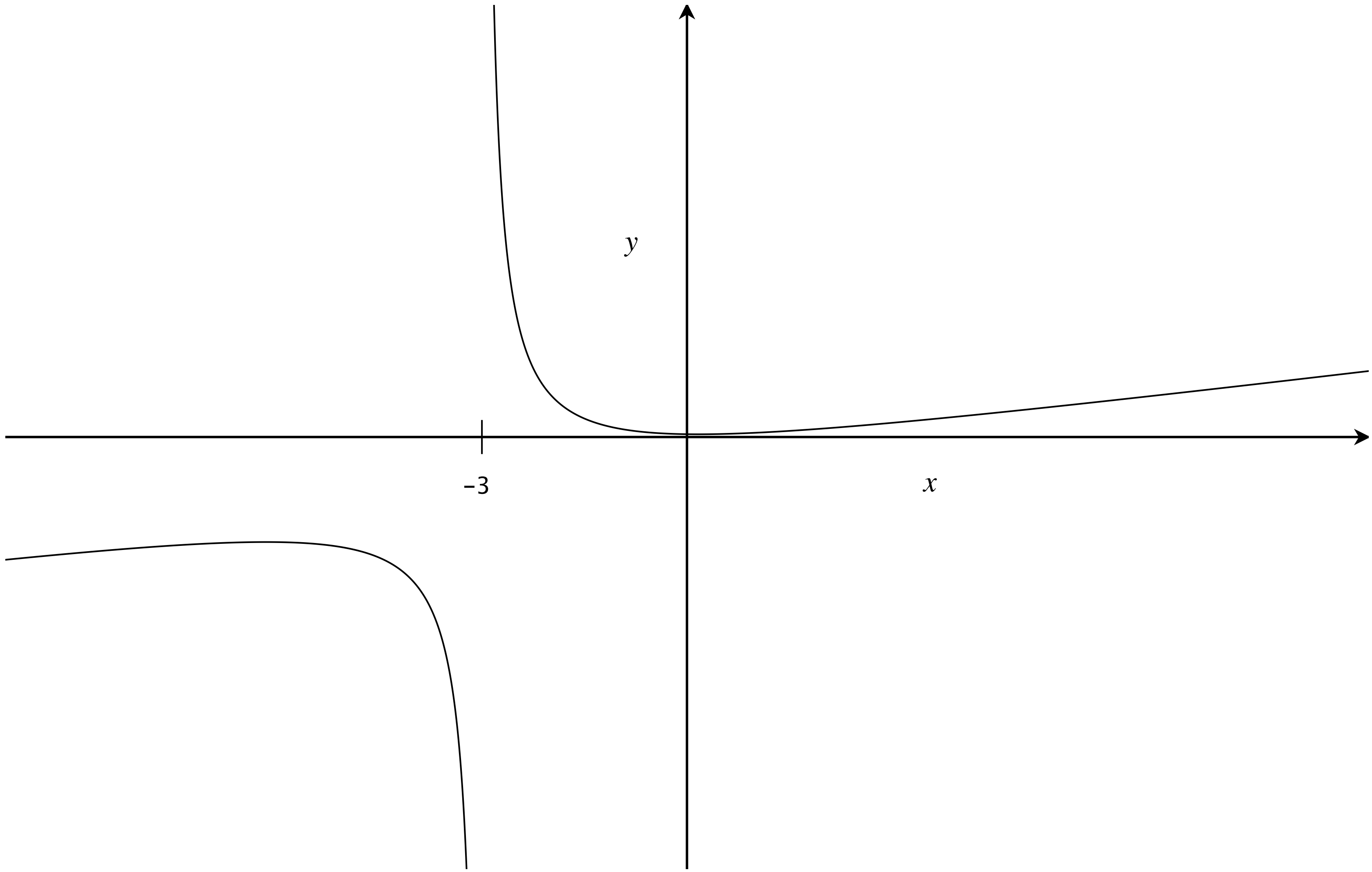
(*) If $x_0 = -3$, then $f(x_0)$ is not defined, so $f(x)$ is not continuous at $x_0 = -3$.

(*) If $x_0 \neq -3$, then $f(x_0)$ is defined and...

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{x^2+1}{x+3} = \frac{\lim\limits_{x \to x_0} x^2+1}{\lim\limits_{x \to x_0} x+3}$$

$$= \frac{\lim\limits_{x \to x_0} x^2 + \lim\limits_{x \to x_0} 1}{\lim\limits_{x \to x_0} x + \lim\limits_{x \to x_0} 3} = \frac{x_0^2+1}{x_0+3} = f(x_0),$$

I.e., $f(x) = \dfrac{x^2+1}{x+3}$ is continuous at every point on the real line except $x = -3$.



(*) Graph of $y = \dfrac{x^2+1}{x+3}$

Continuity of basic functions:

1. Constant functions are continuous on the entire real line.

2. If k is a positive integer, then $f(x) = x^k$ is continuous for all x.

3. If α is any real number, then $f(x) = x^\alpha$ is continuous for all $x > 0$.

4. The function $f(x) = e^x$ is continuous for all x.

5. The function $f(x) = \ln x$ is continuous for all $x > 0$.

Rules for combinations:

6. If $f(x)$ and $g(x)$ are both continuous at $x = x_0$, then $f(x) + g(x)$, $f(x) - g(x)$ and $f(x)g(x)$ are all continuous at x_0. If $g(x_0) \neq 0$, then $f(x)/g(x)$ is also continuous at x_0.

Therefore, if $f(x)$ and $g(x)$ are both continuous in the interval I, then $f(x) + g(x)$, $f(x) - g(x)$ and $f(x)g(x)$ are all continuous in I. Furthermore, $f(x)/g(x)$ is continuous at every point x in I where $g(x) \neq 0$.

7. If $g(x)$ is continuous at $x = x_0$ and $f(x)$ is continuous at $y_0 = g(x_0)$, then $f(g(x))$ is continuous at x_0. Therefore if $g(x)$ is continuous in the interval I and $f(x)$ is continuous in

$$g(I) = \{g(x) : x \in I\}$$

then $f(g(x))$ is continuous in I.

Comment: From time to time, logarithm functions to bases other than e can be useful. For example, computer scientists use $\log_2(x)$ quite a bit, and engineers often use $\log_{10}(x)$. All these different logarithm functions have the same properties as $\ln x$ as far as limits and continuity are concerned *because they are all constant multiples of* $\ln x$, as follows from the change of base formula:

$$\log_b x = \frac{\ln x}{\ln b} = \frac{1}{\ln b} \cdot \ln x.$$

Similarly, it is sometimes convenient to use exponential functions to bases other than the natural base e. Once again, the properties of other exponential functions with regards to limits and continuity are the same as the *natural* exponential function e^x because for any $a > 0$,

$$a^x = e^{(\ln a)x},$$

as you can verify using basic properties of logarithms and exponentials.