Example. If $f(x) = 2x^3 - 4x^2 + 5x + 1$, then

$$f'(x) = 6x^2 - 8x + 5$$

Observation: $f'(x)$ is also a differentiable function...

$$\frac{d}{dx}\big(f'(x)\big) = \frac{d}{dx}\big(6x^2 - 8x + 5\big) = 12x - 8$$

The derivative of $f'(x)$ is called the ***second derivative*** of $f(x)$, and is denoted by $f''(x)$.

Observation: $f''(x)$ is also a differentiable function...

$$\frac{d}{dx}\big(f''(x)\big) = \frac{d}{dx}(12x - 8) = 12$$

The derivative of $f''(x)$ is called the ***third derivative*** of $f(x)$, and is denoted by $f'''(x)$.

And so on...

Higher order derivatives.

*The derivatives of the derivatives (of the derivatives, etc.) of $y = f(x)$ are called the **higher order** derivatives of $f(x)$.*

Notation: We use the following notations interchangeably for derivatives and higher order derivatives of a function $y = f(x)$

(First) derivative: $$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x))$$

Second derivative: $$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(f(x))$$

Third derivative: $$y''' = f'''(x) = \frac{d^3y}{dx^3} = \frac{d^3}{dx^3}(f(x))$$

Fourth derivative: $$y^{(4)} = f^{(4)}(x) = \frac{d^4y}{dx^4} = \frac{d^4}{dx^4}(f(x))$$

$\vdots$ $\qquad\qquad\qquad$ $\vdots$

n^{th} derivative: $$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}(f(x))$$

Observation: For derivatives of order 4 or higher, people generally do not write expressions like

$$f^{\overbrace{\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime}^{17}}(x)$$

For obvious reasons.

Example. Find the third derivative of the function $w = \sqrt{u}$.

First, write $w = u^{1/2}$, then

$$\frac{dw}{du} = \frac{1}{2}u^{-1/2} \quad \Longrightarrow \quad \frac{d^2w}{du^2} = -\frac{1}{4}u^{-3/2} \quad \Longrightarrow \quad \frac{d^3w}{du^3} = \frac{3}{8}u^{-5/2}$$

Example. Find the second derivative of $f(x) = \ln(x^2 + 1)$.

$$f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

$$\Longrightarrow \quad f''(x) = \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

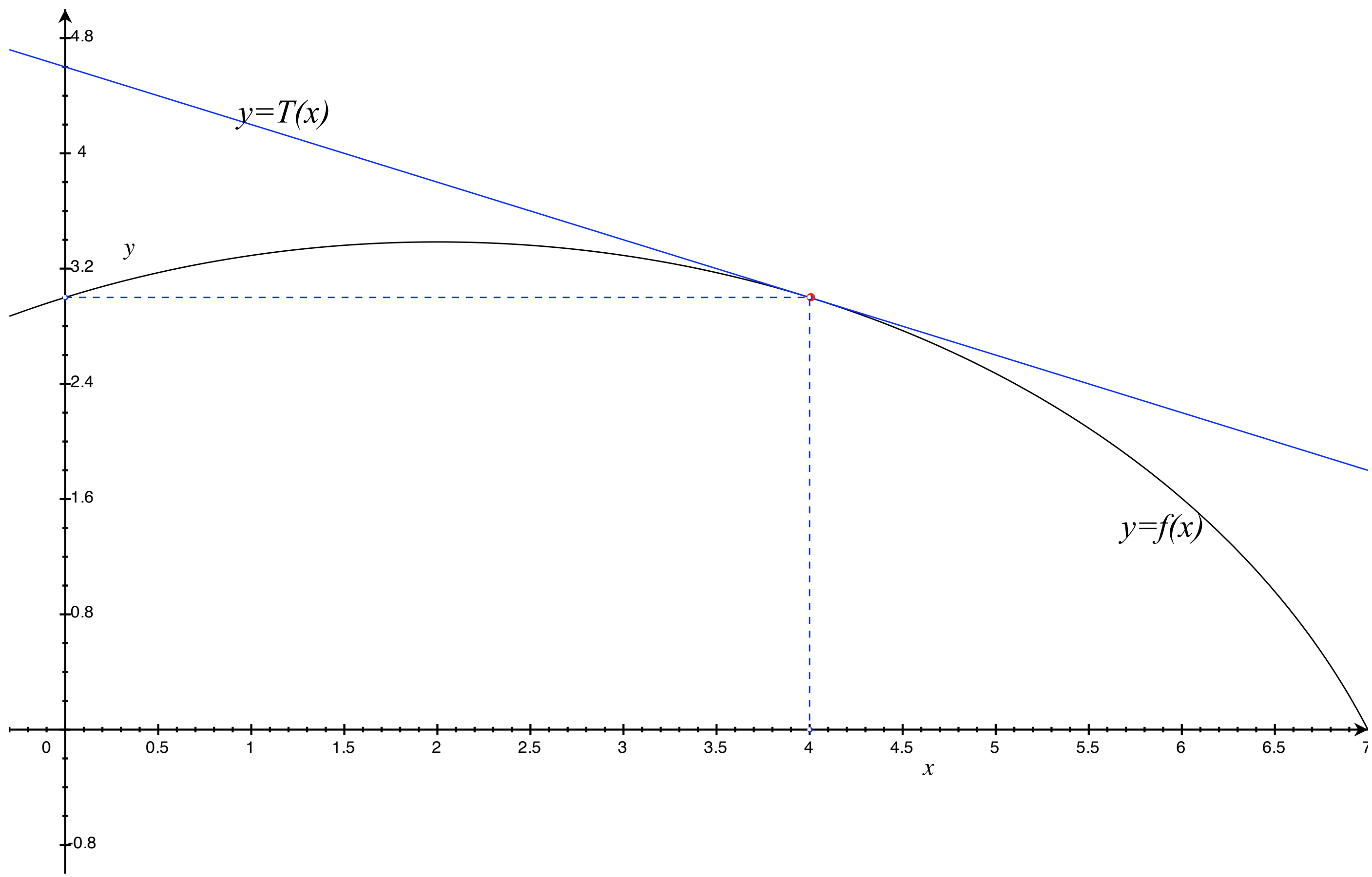
The derivative, $f'(x)$, gives the rate of change of $y = f(x)$ with respect to x, and gives useful information about the function.

What do higher order derivatives tell us?

(*) We will see that the second derivative $f''(x)$ has an observable geometric interpretation, describing an important aspect of the shape of the graph of $y = f(x)$.

(*) Another way we can use higher order derivatives is to improve on *linear approximation.*

Example. The graph of the function $f(x) = \sqrt{-x^2 + 4x + 25} - 2$, and the graph of the linear approximation to this graph at the point $(4, 3)$, $T(x) = 3 - 0.4(x - 4)$, are both displayed in the figure below.



Observations:

1. The linear approximation is good when $x \approx 4$, say $3.5 < x < 4.5$.

2. Linear approximation becomes much less accurate as x moves away from 4.

Why?

*One explanation is that the graph of $y = T(x)$ is a straight line with constant slope, but the slope of $y = f(x)$ is changing, and as x moves away from 4, the graph $y = f(x)$ **bends away** from the graph $y = T(x)$.*

In other words, the linear approximation $f(x) \approx T(x)$ is

- Reasonably good when $x \approx 4$ because $T(4) = f(4)$ and $T'(4) = f'(4)$, but
- less accurate when x moves away from 4 because the slope of $T(x)$ is not *changing* like the slope of $f(x)$

Idea: To improve on linear approximation, find a function $T_2(x)$ with the following properties.

(i) $T_2(4) = f(4)$

(ii) $T_2'(4) = f'(4)$

(iii) $T_2''(4) = f''(4)$

This condition means that the slope of T_2 is changing at the same rate as f when $x = 4$.

(iv) and T_2 should be as 'simple' a function as possible.

(*) A linear function won't work here, because if T_2 is linear, then $T_2'' = 0$ but $f''(4) \neq 0$.

(*) The next simplest type of function is *quadratic*, so we try something like

$$T_2(x) = A + B(x-4) + C(x-4)^2.$$

(Using $(x-4)$ instead of x makes the algebra easier.)

Condition (i):

$$T_2(4) = f(4) \implies A + B(4-4) + C(4-4)^2 = f(4) \implies A = f(4).$$

Condition (ii):

$$T_2'(4) = f(4) \implies B + 2C(x-4) = f'(4) \implies B = f'(4)$$

Condition (iii):

$$T_2''(4) = f''(4) \implies 2C = f''(4) \implies C = \frac{f''(4)}{2}$$

Conclusion: $T_2(x) = f(4) + f'(4)(x-4) + \dfrac{f''(4)}{2}(x-4)^2$

Calculations:

$$f(x) = \left(-x^2 + 4x + 25\right)^{1/2} - 2 \implies f(4) = 3$$

$$f'(x) = \frac{1}{2}\left(-x^2 + 4x + 25\right)^{-1/2} \cdot (-2x+4) = (2-x)\left(-x^2 + 4x + 25\right)^{-1/2}$$

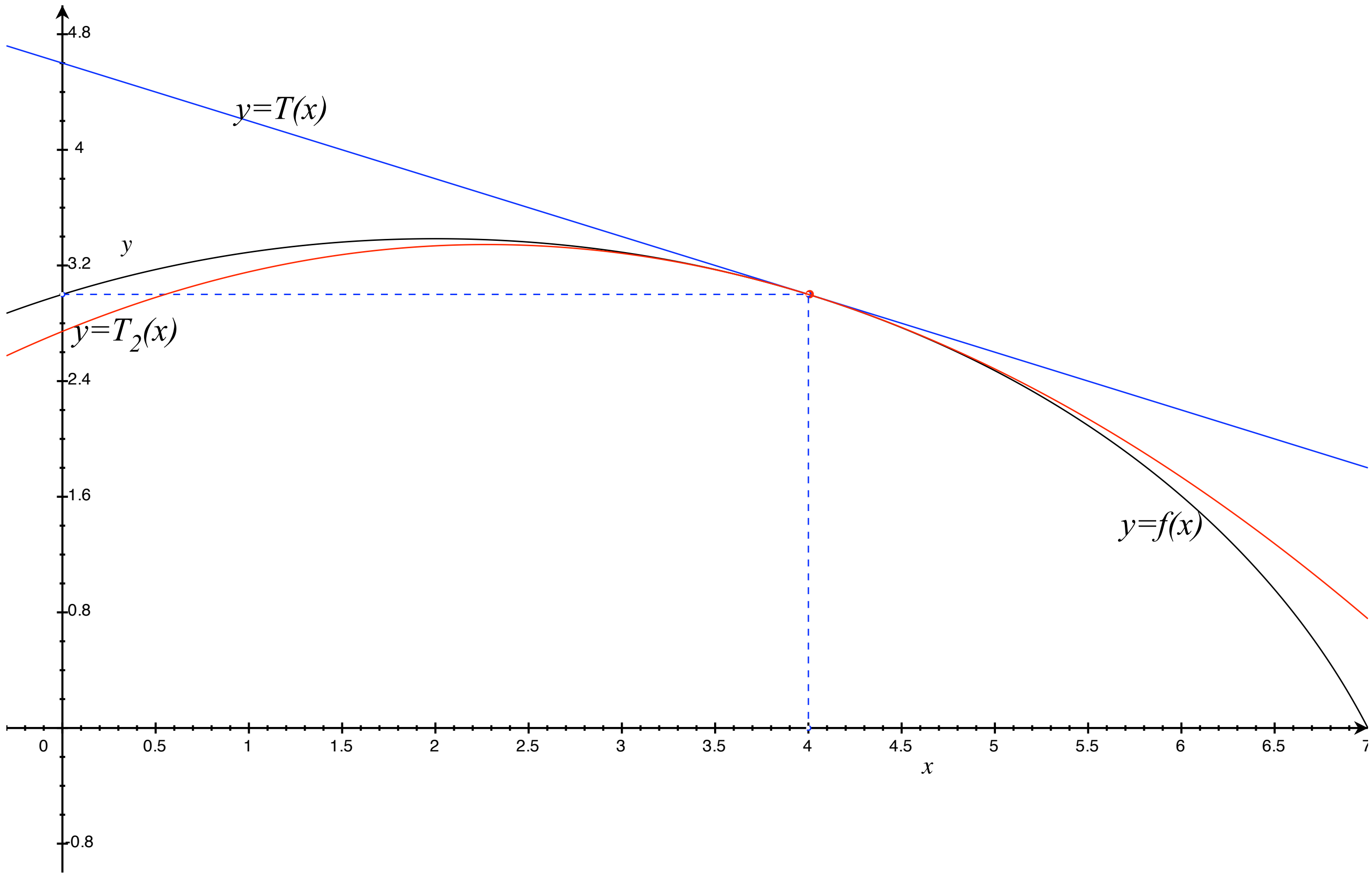
$$\implies f'(4) = -0.4$$

$$\begin{aligned} f''(x) = (-1)\left(-x^2 + 4x + 25\right)^{-1/2} \\ + (2-x)\left(-\frac{1}{2}\right)\left(-x^2 + 4x + 25\right)^{-3/2}(4-2x) \\ = -\left[\left(-x^2 + 4x + 25\right)^{-1/2} + (2-x)^2\left(-x^2 + 4x + 25\right)^{-3/2}\right] \end{aligned}$$

$$\implies f''(4) = -(0.2 + 4 \cdot 0.008) = -0.232$$

Final conclusion:

$$T_2(x) = 3 - 0.4(x-4) - \frac{0.232}{2}(x-4)^2 = 3 - 0.4(x-4) - 0.116(x-4)^2$$



Linear and quadratic approximations to $f(x) = \sqrt{-x^2 + 4x + 25} - 2$.

Definition: The ***quadratic Taylor polynomial*** for the function $y = f(x)$, centered at $(x_0, f(x_0))$, is the function

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

This function has the properties

- $T_2(x_0) = f(x_0)$
- $T_2'(x_0) = f'(x_0)$
- $T_2''(x_0) = f''(x_0)$

Quadratic approximation: *If* $x \approx x_0$, *then* $f(x) \approx T_2(x)$.

Observation: $T_2(x) = T(x) + \dfrac{f''(x_0)}{2}(x - x_0)^2$. I.e., $T_2(x)$ 'corrects' the linear approximation $T(x)$ by adding a quadratic term.

Example. Find the quadratic Taylor polynomial for $f(x) = \sqrt{x}$, centered at $x_0 = 25$.

We need to find $f(25)$, $f'(25)$ and $f''(25)$...

$$f(x) = \sqrt{x} = x^{1/2} \implies f'(x) = \frac{1}{2}x^{-1/2} \text{ and } f''(x) = -\frac{1}{4}x^{-3/2},$$

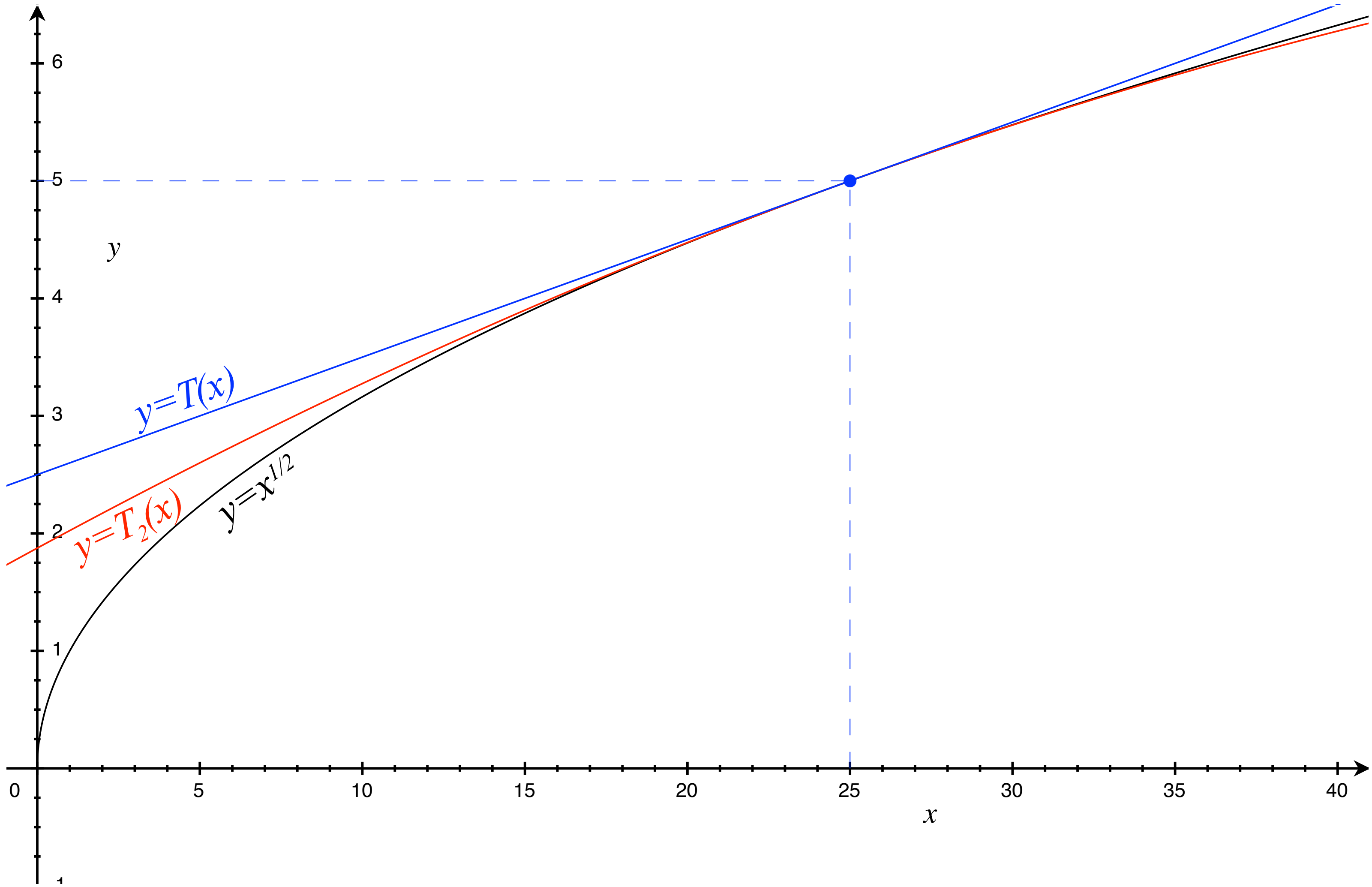
so $f(25) = 25^{1/2} = 5$ and

$$f'(25) = \frac{1}{2}25^{-1/2} = \frac{1}{10} \text{ and } f''(25) = -\frac{1}{4}25^{-3/2} = -\frac{1}{500}$$

So...

$$\begin{aligned} T_2(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \\ &= 5 + \underbrace{\frac{1}{10}}_{f'(25)}(x-25) \underbrace{-\frac{1}{1000}}_{f''(25)/2}(x-25)^2 \end{aligned}$$

Question: How well does quadratic approximation do (a) compared to linear approximation and (b) overall?



Answer 1: *It looks great, based on the pretty picture — better than linear and close overall for* $20 < x < 30$.

Answer 2: *Some numerical comparisons:*

x	$T(x)$	$T_2(x)$	$\sqrt{x}$ (calculator)	$\lvert\sqrt{x} - T(x)\rvert$	$\lvert\sqrt{x} - T_2(x)\rvert$
25	5	5	5	0	0
24	4.9	4.899	$4.898979\ldots$	> 0.001	< 0.000021
26	5.1	5.099	$5.099019\ldots$	> 0.0009	< 0.00002
23	4.8	4.796	$4.795831\ldots$	> 0.004	< 0.00017
27	5.2	5.196	$5.196152\ldots$	> 0.003	< 0.00016
20	4.5	4.475	$4.472135\ldots$	> 0.027	< 0.0029
30	5.5	5.475	$5.477225\ldots$	> 0.022	< 0.0023
16	4.1	4.019	4	0.1	0.019
36	6.1	5.979	6	0.1	0.021