Average elasticity: The average ***elasticity*** of one variable with respect to another is the ratio of their respective *percentage* changes.

Example: If income increases by 5% over a certain period and as result, consumption increases by 4.2%, then the average *income-elasticity* of *consumption* is

$$E_{C/I} = \frac{\%\Delta C}{\%\Delta I} = \frac{4.2\%}{5\%} = 0.84.$$

Definition: If $y = f(x)$, then the average elasticity of y with respect to x (also called the x-elasticity of y) over the interval $(x, x + \Delta x)$ is

$$E_{y/x} = \frac{\%\Delta y}{\%\Delta x} = \frac{\frac{\Delta y}{y} \cdot 100\%}{\frac{\Delta x}{x} \cdot 100\%} = \frac{\left(\frac{f(x+\Delta x) - f(x)}{f(x)}\right) \cdot 100\%}{\frac{\Delta x}{x} \cdot 100\%}$$

Observation: The elasticity is *units-free*—a pure number—because the units for x cancel each other out in the denominator and the units for y cancel in the numerator.

The formula for elasticity simplifies:

$$E_{y/x} = \frac{\%\Delta y}{\%\Delta x} = \frac{\dfrac{\Delta y}{y_0} \cdot \cancel{100\%}}{\dfrac{\Delta x}{x_0} \cdot \cancel{100\%}} = \frac{\Delta y}{\Delta x} \cdot \frac{x_0}{y_0}.$$

Example: Suppose that $y = 0.3x^2$. What is $E_{y/x}$ in the interval $(1, 3)$?

In this case $x_0 = 1$ and $\Delta x = 3 - 1 = 2$, and $y_0 = y(1) = 0.3$ and $\Delta y = y(3) - y(1) = 2.7 - 0.3 = 2.4$. Therefore

$$E_{y/x} = \frac{2.4}{2} \cdot \frac{1}{0.3} = 4.$$

By what percentage does y increase in this interval? Two ways to answer...

$$\%\Delta y = \frac{\Delta y}{y} \cdot 100\% = \frac{2.4}{0.3} \cdot 100\% = 800\%,$$

or

$$\%\Delta y = E_{y/x} \cdot \%\Delta x = 4 \cdot \frac{\Delta x}{x} \cdot 100\% = 4 \cdot 200\% = 800\%.$$

Definition: The ***point-elasticity*** of y with respect to x is the *limit of the average elasticity as* $\Delta x \to 0$. We denote point-elasticity by the Greek letter η (eta), so

$$\eta_{y/x} = \lim_{\Delta x \to 0} E_{y/x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right) = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right) \cdot \frac{x}{y} = \frac{dy}{dx} \cdot \frac{x}{y}.$$

Observation: As defined, if $y = f(x)$, then $\eta_{y/x}$ is also a function of x:

$$\eta_{y/x} = \frac{dy}{dx} \cdot \frac{x}{y} = x \cdot \frac{f'(x)}{f(x)}.$$

Example. Find $\eta_{y/x}$ if $y = 2x + 3$...

$$\eta_{y/x} = x \cdot \frac{2}{2x+3} = \frac{2x}{2x+3}.$$

Example. Find $\eta_{v/u}$ if $v = 3\sqrt{u}$... $= 3u^{1/2}$

$$\eta_{v/u} = \frac{dv}{du} \cdot \frac{u}{v} = \frac{\cancel{3}}{2} u^{-1/2} \cdot \frac{u}{\cancel{3} u^{1/2}} = \frac{1}{2} \cdot \frac{\cancel{u^{1/2}}}{\cancel{u^{1/2}}} = \frac{1}{2}.$$

Price-elasticity of demand.

Example: If $q = 300 - 1.2p$ is the demand function for a firm, then

$$\eta_{q/p} = (300 - 1.2p)' \cdot \frac{p}{300 - 1.2p} = -1.2 \cdot \frac{p}{300 - 1.2p} = -\frac{1.2p}{300 - 1.2p}$$

is the price-elasticity of demand for the firm's product, *expressed as a function of* p.

What is the price-elasticity of demand when the price is $p = 200$?

$$\eta_{q/p}\Big|_{p=200} = -\frac{240}{300 - 240} = -4.$$

Question: Demand equations are often written in the form $p = f(q)$, with the demand q as the independent variable and the price p as the dependent variable.

How can we compute the price-elasticity of demand in such cases?

Answer: we return to the definition...

$$\eta_{q/p} = \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} \cdot \frac{p}{q}$$

and make two observations...

1. If $\Delta p \to 0$ then $\Delta q \to 0$ and *vice versa*, and

2. $\dfrac{\Delta q}{\Delta p} = \dfrac{1}{\frac{\Delta p}{\Delta q}}$, so

$$\lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \lim_{\Delta q \to 0} \frac{1}{\dfrac{\Delta p}{\Delta q}} = \frac{1}{\lim_{\Delta q \to 0} \dfrac{\Delta p}{\Delta q}} = \frac{1}{dp/dq}.$$

Plugging this back into the definition of elasticity gives

$$\eta_{q/p} = \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} \cdot \frac{p}{q} = \frac{1}{dp/dq} \cdot \frac{p}{q} = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}}$$

Example: Suppose that the demand equation for a firm's product is given by $p = 250 - \frac{5}{6}q$. What is the price-elasticity of demand for the firm's product when $p = 200$?

In this case we first find the elasticity as a function of q...

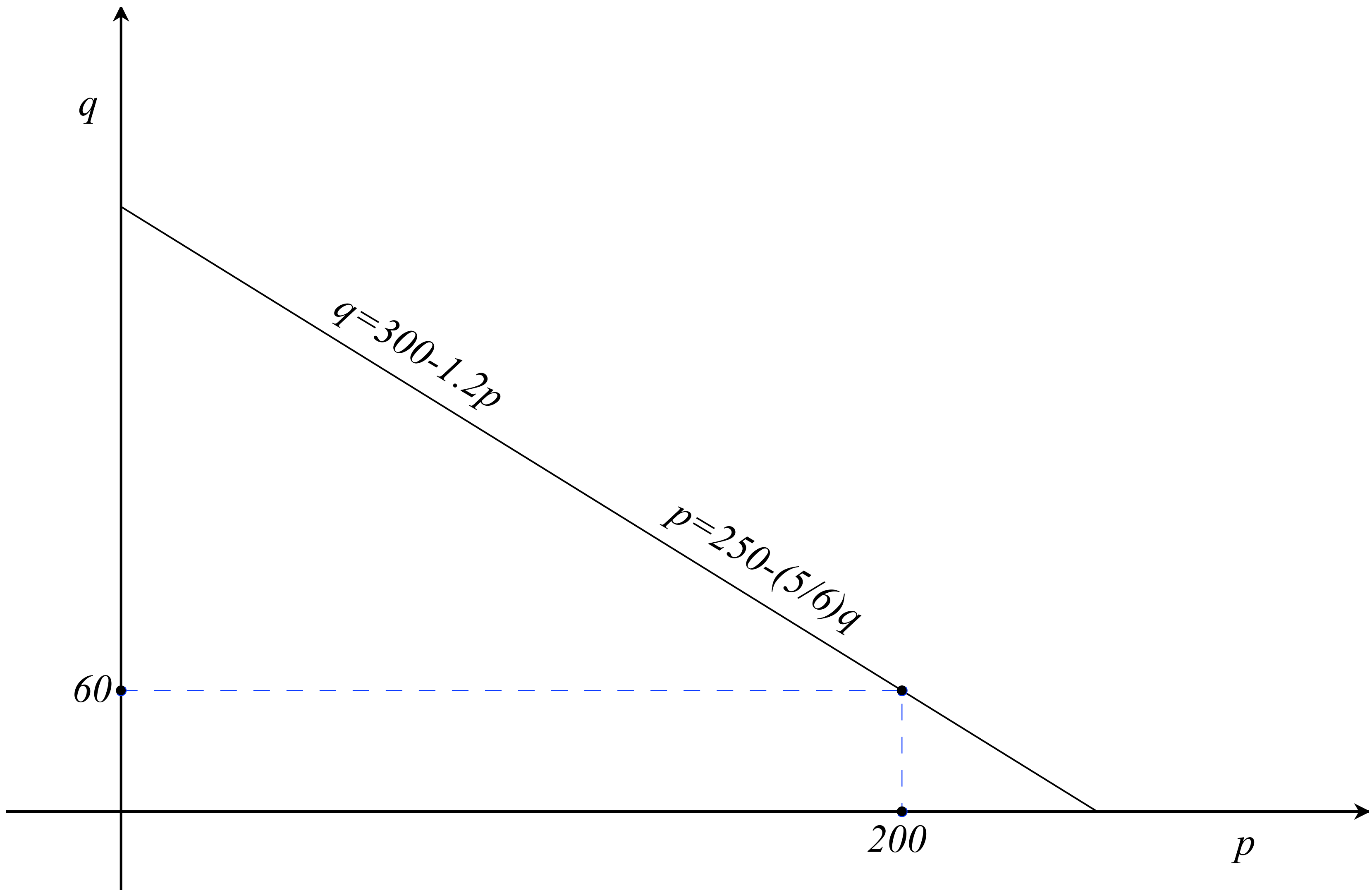
$$\eta_{q/p} = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{250 - \frac{5}{6}q}{q}}{-\frac{5}{6}} = -\frac{300 - q}{q}$$

and then find q when $p = 200$...

$$p = 200 \implies 200 = 250 - \frac{5}{6}q \implies \frac{5}{6}q = 50 \implies q = 60$$

and then compute the elasticity

$$\eta_{q/p}\Big|_{p=200} = \eta_{q/p}\Big|_{q=60} = -\frac{300 - 60}{60} = -4$$



(*) Price-elasticity of demand depends on the point on the demand curve, not on the particular equation that we use to describe the demand curve.

Linear approximation for percentage change.

By definition, $\eta_{y/x} = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x}$.

This means: if Δx is small, then $\eta_{y/x} \approx \frac{\%\Delta y}{\%\Delta x}$

and therefore, if Δx is small, then

$$\%\Delta y \approx \eta_{y/x} \cdot (\%\Delta x)\,.$$

Example: Find $\eta_{y/x}$ if $y = 2x^{0.4}$...

$$\eta_{y/x} = \frac{dy}{dx} \cdot \frac{x}{y} = 0.8x^{-0.6} \cdot \frac{x}{2x^{0.4}} = 0.4.$$

What is $\%\Delta y$ if x increases from 10 to 10.3?

The percentage change in x is $\%\Delta x = (0.3/10) \cdot 100\% = 3\%$, and linear approximation gives $\%\Delta y \approx \eta_{y/x} \cdot \%\Delta x = 0.4 \cdot 3\% = 1.2\%$.

(The actual percentage change in y is $\frac{2 \cdot 10.3^{0.4} - 2 \cdot 10^{0.4}}{2 \cdot 10^{0.4}} \cdot 100\% \approx 1.19\%$)

Classifying price-elasticity of demand.

For a normal good, the price elasticity of demand (as defined here and in our textbook) is always negative, because

$$\eta_{q/p} = \frac{dq}{dp} \cdot \frac{p}{q} \quad \text{or} \quad \eta_{q/p} = \frac{p/q}{dp/dq}$$

and p and q are both ***positive***, while both dq/dp and dp/dq are ***negative***.

This makes sense because for a normal good, if the price rises then the demand will go down and vice versa, so if $\%\Delta p > 0$, we expect $\%\Delta q < 0$ and vice versa.

Definition:

- If $\eta_{q/p} < -1$ (so $|\eta_{q/p}| > 1$), demand is said to be ***elastic***.
- If $0 > \eta_{q/p} > -1$ (so $|\eta_{q/p}| < 1$), demand is said to be ***inelastic***.
- If $\eta_{q/p} = -1$ (so $|\eta_{q/p}| = 1$), demand is said to have ***unit elasticity*** (or to be unit-elastic).

When demand is *elastic* a 1% percentage change in price will result in a larger than 1% change in demand (in the opposite direction).

When demand is *inelastic*, a 1% percentage change in price will result in a smaller than 1% change in demand (again, in the opposite direction).

Intuition: If demand is *elastic*, then *lowering* the price of a good will result in higher revenue. And if the demand is *inelastic*, then *raising* the price of a good will result in higher revenue.

To verify this intuition, we study the relation between *price-elasticity of demand* and *marginal revenue.* Using the product rule, we see that

$$\frac{dr}{dq} = \frac{d}{dq}\overbrace{(pq)}^{r} = \frac{dp}{dq} \cdot q + p \cdot \frac{dq}{dq} = p + q\frac{dp}{dq} = p\left(1 + \frac{q}{p} \cdot \frac{dp}{dq}\right)$$

Now we observe that

$$\frac{q}{p} \cdot \frac{dp}{dq} = \frac{1}{\frac{p}{q}} \cdot \frac{dp}{dq} = \frac{\frac{dp}{dq}}{\frac{p}{q}} = \frac{1}{\eta} \ldots \quad \text{so} \quad \boxed{\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right)}$$

Therefore:

1. If $\eta < -1$ (elastic demand), then $0 > \frac{1}{\eta} > -1$ so $\dfrac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right) > 0$. In this case, *lowering* p will *raise* q and since $dr/dq > 0$, this will *increase revenue.*

2. If $0 > \eta > -1$ (inelastic demand), then $\frac{1}{\eta} < -1$ so $\dfrac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right) < 0$. In this case, *raising* p will *lower* q and since $dr/dq < 0$, this will *increase* revenue.