The Chain Rule:

$$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x).$$

Example 1. Find the derivative of $y = \sqrt{x^3 + x + 2}$.

(*) $\sqrt{x^3 + x + 2} = f(g(x))$ where $f(u) = \sqrt{u} = u^{1/2}$ and $g(x) = x^3 + x + 2$, so...

$$\frac{d}{dx}\sqrt{x^3 + x + 2} = \frac{d}{dx}\left(x^3 + x + 2\right)^{1/2} = \overbrace{\frac{1}{2}\left(x^3 + x + 2\right)^{-1/2}}^{f'(g(x))} \cdot \underbrace{(3x^2 + 1)}_{g'(x)}$$

(*) The chain rule can also be expressed as follows. If $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$ and

$$\frac{dy}{dx} = \overbrace{\frac{dy}{du}}^{f'(g(x))} \cdot \overbrace{\frac{du}{dx}}^{g'(x)}$$

Explanation: With $y = f(u)$ and $u = g(x)$, *linear approximation* says that if $\Delta x \approx 0$, then

$$\Delta u = g(x + \Delta x) - g(x) \approx g'(x)\Delta x. \tag{1}$$

Likewise, if $\Delta u \approx 0$, then

$$\Delta y = f(u + \Delta u) - f(u) \approx f'(u)\Delta u. \tag{2}$$

Now, if Δx is close to 0, then ***so is*** Δu (because $g'(x)$ is fixed), so if $\Delta x \approx 0$, then using both approximations (1) and (2) gives

$$\Delta y \approx f'(u)\Delta u \approx f'(u)\overbrace{g'(x)\Delta x}^{\approx \Delta u} = f'(g(x))g'(x)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(g(x))g'(x)\cancel{\Delta x}}{\cancel{\Delta x}} = f'(g(x))g'(x).$$

These approximations all becomes more and more accurate as $\Delta x \to 0$, and therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(g(x))g'(x).$$

Example 2. Find the point(s) on the graph of $h(x) = \left(x^2 - 3x - 4\right)^3$ where the tangent line is horizontal.

(*) We need to find the point(s) where $h'(x) = 0$, which means that we first have to differentiate $h(x)$.

(*) Differentiation: use the chain rule on $h(x) = f(g(x))$, with $f(u) = u^3$ and $g(x) = x^2 - 3x - 4$:

$$h'(x) = \underbrace{3(\overbrace{x^2 - 3x - 4}^{u})^2}_{df/du}(\overbrace{2x - 3}^{dg/dx})$$

(*) **Recall:** A product $AB = 0$ if and only if $A = 0$ or $B = 0$, so

$$h'(x) = 0 \Longleftrightarrow x^2 - 3x - 4 = 0 \;\text{ or }\; 2x - 3 = 0$$

and $x^2 - 3x - 4 = (x - 4)(x + 1)$, so $h'(x) = 0$ when $x = -1$, $x = 3/2$ and $x = 4$.

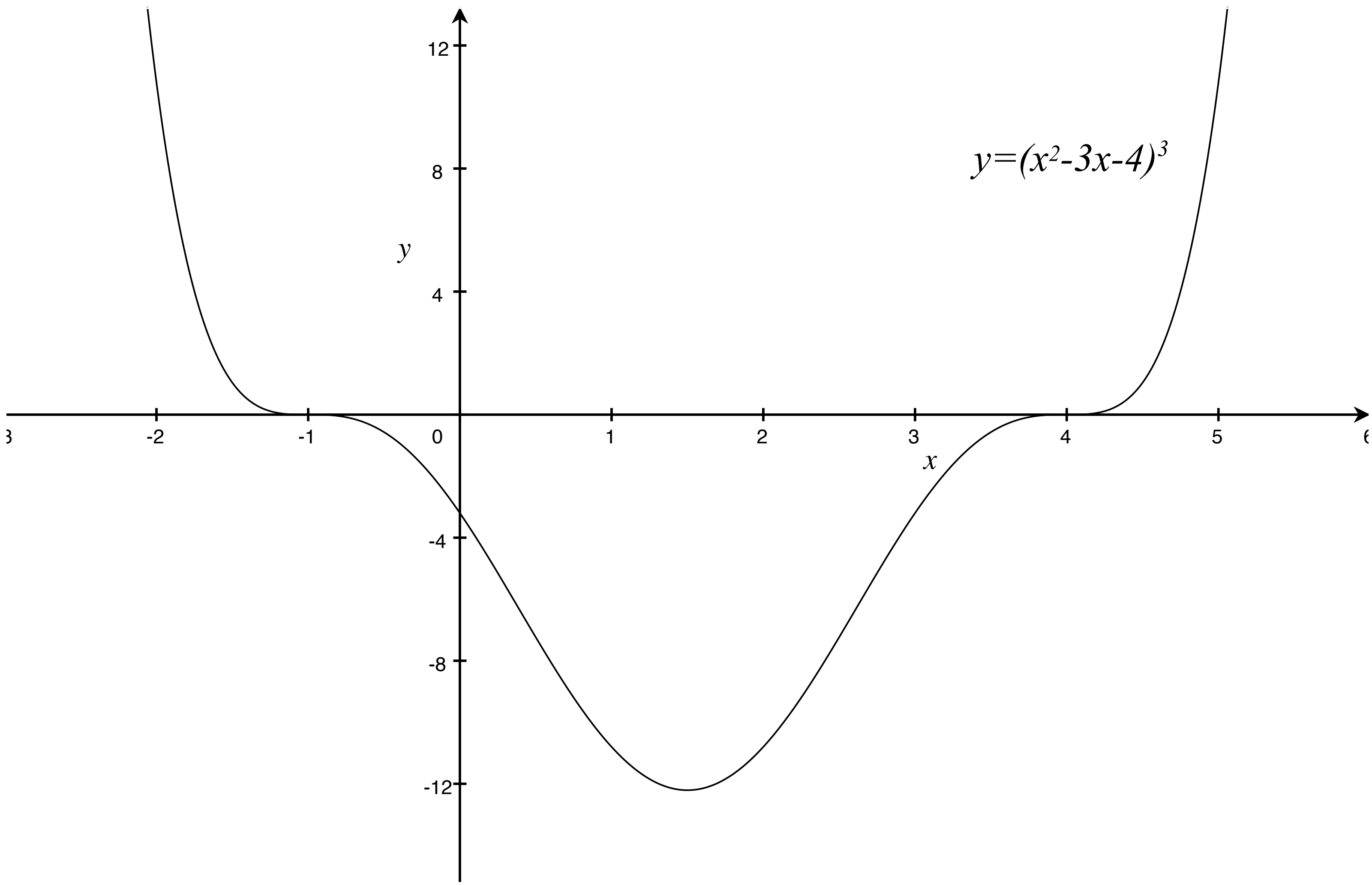

Figure 1: The graph of $y = (x^2 - 3x - 4)^3$.

Differentiating without the chain rule...

$$\begin{aligned} h(x) = (x^2-3x-4)^3 &= (x^2-3x-4)^2(x^2-3x-4) \\ &= (x^4-6x^3+x^2+24x+16)(x^2-3x-4) \\ &= x^6-9x^5+15x^4+45x^3-60x^2-144x-64 \end{aligned}$$

So

$$h'(x) = 6x^5-45x^4+60x^3+135x^2-120x-144.$$

Now all we have to do is to solve the equation

$$6x^5-45x^4+60x^3+135x^2-120x-144 = 0...$$

Observation: Using the chain rule in this example has two advantages:

(*) No messy arithmetic.

(*) The chain rule gives $h'(x)$ in a (partially) factored form, which makes solving the equation $h'(x) = 0$ is *much* easier.

Example 3. Find the equation of the tangent line to the graph

$$y = \frac{2}{\sqrt[3]{x^2 + 4}}$$

at the point $(2, 1)$.

We can use the quotient rule combined with the chain rule to find the derivative dy/dx, or we can just use the chain rule and the observation that

$$y = \frac{2}{\sqrt[3]{x^2+4}} = 2(x^2+4)^{-1/3}$$

$$\implies \frac{dy}{dx} = 2 \cdot \left(-\frac{1}{3}\right)(x^2+4)^{-4/3} \cdot (2x) = -\frac{4x}{3}(x^2+4)^{-4/3}$$

$$\implies \left.\frac{dy}{dx}\right|_{x=2} = -\frac{8}{3} \cdot 8^{-4/3} = -\frac{1}{6}.$$

Now we use the point-slope formula to find the equation of the tangent line:

$$y - 1 = -\frac{1}{6}(x-2) \implies y = 1 - \frac{1}{6}(x-2) \qquad \left(\text{or } y = \frac{4}{3} - \frac{x}{6}\right)$$

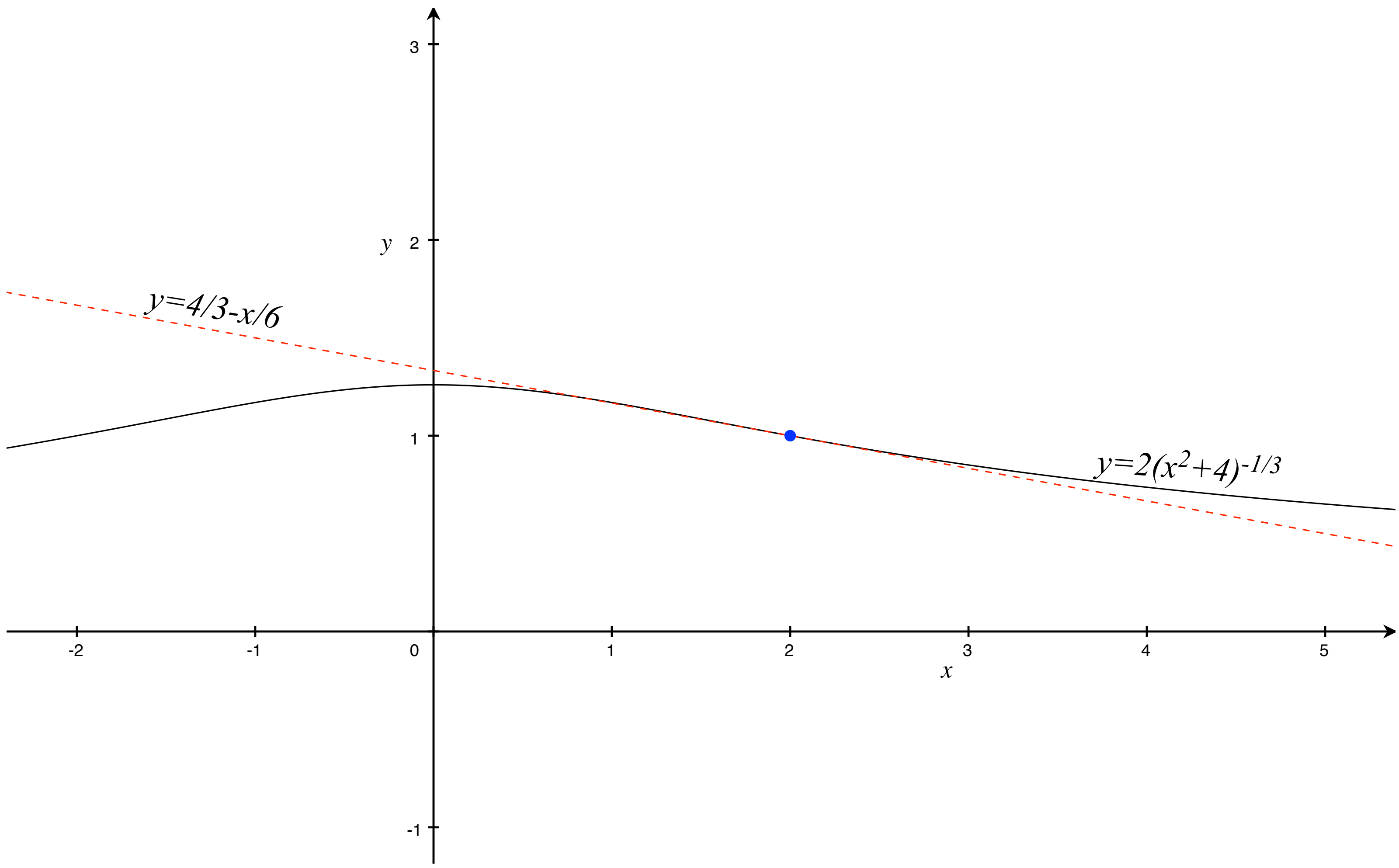

Figure 2: The graphs of $y = 2(x^2 + 4)^{-1/3}$ and the tangent line at $(2, 1)$.

Observation: $f(x)/g(x) = f(x)g(x)^{-1}$, so...

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)g(x)^{-1}\right)$$

$$= f'(x)g(x)^{-1} + f(x)\frac{d}{dx}\left(g(x)^{-1}\right)$$

$$= f'(x)g(x)^{-1} + f(x)\left((-1)g(x)^{-2}g'(x)\right)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

I.e., the quotient rule follows from combining the product rule and chain rule.

Example 4. Find the derivative of $f(x) = 3x\sqrt{x^2+1}$.

$$\begin{aligned}
f'(x) &= \overbrace{(3x)'(x^2+1)^{1/2} + 3x\left((x^2+1)^{1/2}\right)'}^{\text{product rule}} \\
&= 3(x^2+1)^{1/2} + 3x\underbrace{\left(\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x\right)}_{\text{chain rule}} \\
&= 3(x^2+1)^{1/2} + 3x^2(x^2+1)^{-1/2} \\
&= \frac{6x^2+3}{\sqrt{x^2+1}}
\end{aligned}$$

Marginal revenue product.

(*) A firm's revenue function is $r = f(q)$ (where q is output).

(*) The firm's production function is $q = g(l)$ (where l is labor input).

(*) It follows that revenue is a function of labor input:

$$r = f(g(l)).$$

(*) The derivative dr/dl is called the firm's *marginal revenue product.*

(*) By the chain rule

$$\frac{dr}{dl} = \frac{dr}{dq} \cdot \frac{dq}{dl},$$

i.e.,

$$\text{marginal revenue product } = (\text{marginal revenue}) \times (\text{marginal product}).$$

Example 5: The revenue function for a firm's product is

$$r = 20q - 0.4q^2$$

and the firm's production function is

$$q = 5\sqrt{3\ell - 14}.$$

(*) Monthly revenue, r, is measured in \$1000s.

(*) Monthly output, q, is measured in 1000s of units.

(*) Labor input, ℓ, is measured in \$1000s per week

(*) Current labor input: $\ell_0 = 10$.

Firm is considering the hiring of a widget polisher who will cost (wages, benefits and taxes) \$500 a week. How will this affect their bottom line?

(*) Current output and revenue: $q_0 = 5\sqrt{30 - 14} = 20$

and $r_0 = 20 \cdot 20 - 0.4 \cdot 20^2 = 240$

(i) An increase of \$500/week in labor cost means $\Delta\ell = 0.5$

(ii) Approximate change in output:

$$\begin{aligned}
\Delta q &\approx \left.\frac{dq}{d\ell}\right|_{\ell_0=10} \cdot \Delta\ell \\
&= \frac{d}{d\ell}\left.\left(5(3\ell-14)^{1/2}\right)\right|_{\ell_0=10} \cdot (0.5) \\
&= \left.\left(5 \cdot \frac{1}{2}(3\ell-14)^{-1/2} \cdot 3\right)\right|_{\ell_0=10} \cdot (0.5) = \frac{15}{16} \quad (= 0.9375).
\end{aligned}$$

(iii) Approximate change in revenue:

$$\begin{aligned}
\Delta r &\approx \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \overbrace{\Delta q}^{\approx 15/16} \\
&\approx \frac{d}{dq}\left.\left(20q - 0.4q^2\right)\right|_{q_0=20} \cdot \frac{15}{16} \\
&= \left.(20 - 0.8q)\right|_{q_0=20} \cdot \frac{15}{16} = \frac{15}{4} = 3.75
\end{aligned}$$

Conclusion: Monthly revenue will increase by about \$234 while monthly costs will increase by \$2000 = \$500 × 4. Firm's profit will decrease by about \$1766.

Where is the chain rule?

$$\Delta r \approx \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \Delta q \approx \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \overbrace{\left.\frac{dq}{d\ell}\right|_{\ell_0=10} \cdot \Delta \ell}^{\approx \Delta q} = \overbrace{\left.\frac{dr}{dq}\right|_{q_0=20} \cdot \left.\frac{dq}{d\ell}\right|_{\ell_0=10}}^{\left.\frac{dr}{d\ell}\right|_{\ell_0=10}} \cdot \Delta \ell$$